The DoF of the 3-user $(p, p+1)$ MIMO Interference Channel

Marc Torrellas, Adrian Agustin, Josep Vidal and Olga Muñoz

***Abstract*—The *degrees of freedom* (DoF) of the 3-user multiple-input multiple-output (MIMO) interference channel (IC) with full channel state information and constant channel coefficients are investigated when $(p, p+1)$ antennas are deployed at the transmitters and receivers, respectively. The point of departure of this paper is the work of Wang *et al.* , which conjectured but not proved the DoF for the antenna settings with $p > 1$. Here the achievability of the DoF outer bound is formally proved using linear methods, thereby avoiding the use of the rational dimensions framework. The proposed transmission scheme exploits asymmetric complex signaling together with symbol extensions in time and space interference alignment concepts. While the paper deals with the $p = 2, 3, \ldots, 6$ cases, providing the specific transmit and receive filters, there are also provided the tools needed for proving the achievability of the optimal DoF for $p > 6$, whose DoF characterization is conjectured.**

***Index Terms*—interference channels, MIMO, interference alignment, degrees of freedom**

I. Introduction

In recent years, the *degrees of freedom* (DoF) have emerged as one of the most important metrics for characterizing interference networks. The DoF describe how the system sum rate scales with the logarithm of the signal-to-noise ratio (SNR) at the high SNR regime. Therefore, the study of the DoF reveals how deploying additional antennas at the nodes (and/or considering multiple time/frequency transmissions) provides additional signal dimensions that can be exploited for data-rate gains. The interference alignment (IA) concept has been one of the fundamental tools to elucidate the optimal DoF for certain configurations of interference networks. The main purpose of IA is to design the transmit filters in such a way that each receiver observes all the interference signals overlapped in a common subspace. The concept was originally proposed in the context of index coding in [1], while it crystallized later on for the 2-user multiple-input multiple-output (MIMO) X-channel in [2] and for the K-user single-input single-output (SISO) interference channel (IC) with $K > 2$ in [3]. Surprisingly, Cadambe and Jafar [3] proposed a linear precoding/decoding scheme that provides *each user half the cake*, and therefore a total of $K/2$ DoF over the network. Additionally, the authors showed that this result generalizes to the MIMO case, obtaining $KM/2$ total DoF when all nodes are equipped with M antennas. For both cases, the achievability of fractional DoF relies on transmitting along an arbitrarily large number of channel uses on a time-varying or frequency-selective channel. However, it fails when considering a constant SISO channel. This is because the equivalent channel matrices result on scaled identity matrices with not enough diversity as compared to the case where channel variations or multiple antennas can be exploited. This limitation revealed that when the channel coefficients are constant the signal dimensions provided by deploying additional antennas (referred in [4] as space extensions) provide more diversity than the signal dimensions exploited through time/frequency extensions.

There is a large number of works in the literature that have employed the IA concept for analyzing the MIMO IC in terms of DoF, see for example [4], [5], [6], [7], [8], [9], [10], [11]. Especially interesting is the work in [9], where the authors showed the DoF reciprocity concept in wireless networks. This property states that given a network with one particular antenna setting, its reciprocal setting, i.e. a network where the number of antennas at the transmitters and receivers (or the roles of transmitters and receivers) are exchanged, has exactly the same DoF. This important result alleviate the challenge of characterizing the channel DoF,

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because only half of the number of antenna settings have to be considered.

After the disrupting idea of IA, different types of IA have been proposed. In the literature, there are two different frameworks for developing IA-based transmit precoders: lattice level IA [12] (*lattice alignment*), and space level IA [2], (*space alignment*). These two techniques arise from the choice between structured or random codes, respectively. Lattice alignment-based techniques use structured coding, e.g. lattice codes, to align the interference on the signal scale level. This idea was exploited in the context of the rational dimensions framework by Motahari *et al.* in [13]. Following this line of research, Ghasemi *et al.* [14] showed that the DoF outer bound may be attained for almost any user and antenna settings. Nevertheless, its rate performance is extremely degraded at medium SNR values [15]. In contrast, space alignment techniques provide a better rate performance at moderate SNR regimes, but they only attain optimality for certain antenna configurations when the channel coefficients are constant.

This paper considers the conventional IA approach, denoted as *linear IA*, under the space alignment framework, to investigate the *linear DoF*, i.e. the DoF that can be achieved using linear encoding strategies, of the 3-user MIMO IC with constant channel coefficients. Two other types of IA have been proposed in the literature under the space alignment framework: ergodic IA (EIA) [16] and opportunistic IA (OIA) [17], [18]. These two approaches do not apply to the scenario considered in this work, as we discuss next. First, EIA relies on repeating the same transmission along two time slots with complementary channel states, i.e. two channel realizations such that the interference is canceled by simply summing up the signals received from both time slots. The surprising result in [16] was that the optimal $\frac{K}{2}$ DoF value can be attained in a K-user SISO IC. However, time-varying channels is a fundamental feature for applying EIA, which differs from the constant channel condition assumed here. Second, OIA exploits the user dimension through scheduling. The idea is to combine the benefits of opportunistic beamforming and IA, which allows a significant reduction of the control plane information sharing. This work assumes a more generic case where users cannot be selected and its channels are given, thus OIA will not be considered.

The most relevant current results in terms of DoF for the 3-user MIMO IC under the linear IA approach are reviewed in the sequel. First, for the SISO case, the best known inner bound was proposed by Cadambe *et al.* in [10]. The authors proposed a linear scheme able to achieve 1.2 DoF, thanks to the *asymmetric complex signaling* concept. This approach, together with symbol extensions in time, is able to independently deal with the real and imaginary components of the channel. As a result, the equivalent channel matrices are no longer scaled identity matrices but present a more sophisticated structure that can be exploited by the IA scheme. This tool has recently been shown to be useful also for the 4-user SISO IC in [19].

For the MIMO case, Wang *et al.* characterized the 3-user MIMO IC [4] in terms of DoF. On the one hand, the DoF outer bound was derived by introducing the change of basis operation, which allows to write the equivalent channels in such a way that the appropriate genie signals to be provided to each receiver can be more easily identified[1]. On the other hand, the proposed DoF inner bound flows from the *subspace alignment chains* concept. This approach proposes a linear transmitter design intertwined among users through the alignment, being optimal in terms of DoF for almost all antenna settings. Nevertheless, it is not known if the DoF outer bound for the SISO and all $(p+1,p)$ and $(p,p+1)$ MIMO settings with $p>1$[2] can be attained using linear schemes when the channel coefficients are constant, and no spatial extensions are allowed, see Section VIII.C in [4]. It is worth pointing out that the optimality of the case $(p,p{+}1)$ has been claimed in [21] by means of asymmetric complex signaling and *subspace alignment chain* concepts, but the result is just sustained on numerical experiments. Therefore, to the best of the authors' knowledge, there is not a formal proof in the literature. Finally, we remark that the information theoretical DoF outer bound, which is independent of the encoding/decoding strategy, has been attained using lattice alignment under the rational dimensions framework.

[1] Interestingly, the change of basis operation has been found to be useful for other settings, e.g. the MIMO rank-deficient IC [20].

[2] The case $p=1$ was previously addressed in [7].

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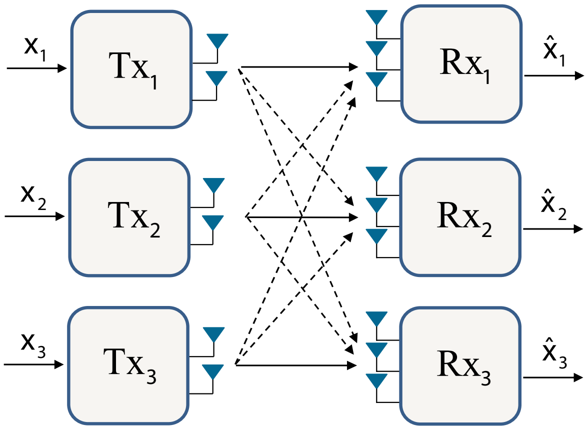

Fig. 1. The 3-user $(2,3)$ MIMO IC. Solid lines define the intended signals, while dotted lines denote the interference signals.

A. Contributions

The goal of this work is to formally prove that the DoF that can be achieved using linear transmit-receive filters, hereafter denoted as *linear DoF*, coincide with the DoF outer bound for the 3-user $(p, p+1)$ MIMO IC even assuming constant channel coefficients. As an example of such scenario, Fig.1 shows the 3-user $(2,3)$ MIMO IC. The proposed scheme is based on interference alignment, *symbol extensions in time* and *asymmetric complex signaling*. Three contributions summarize this work:

- We prove that the 3-user MIMO IC with constant channel coefficients has exactly $\frac{p(p+1)}{2p+1}$ linear DoF per user for $p = 2, 3...6$, see Theorem 1 and Theorem 2 in Sections V and VI, respectively.
- The proposed transmit precoding matrices present a specific structure that can be generalized for any value of p. This structure is characterized by two properties: *i)* there are some elements equal to zero, and *ii)* all transmit precoders are defined as a function of 3 matrices, denoted as the *support precoding blocks*. An iterative algorithm is proposed, able to find the structure of each precoding matrix for any value of p.
- The proof methodology is also generalized. Based on this, we conjecture that the value $\frac{p(p+1)}{2p+1}$ corresponds also to the linear DoF for any $p \geq 7$.

B. Organization

The paper is organized as follows. Section II introduces the system model considered. Next, Section III reviews the DoF for our specific scenario, as well as DoF achievability conditions when using IA. The structure of our precoding scheme is defined in Section IV, as well as how this structure is related to the alignment chains. Section V is devoted to the $p = 2$ case, while Section VI addresses the $p = 3$ case, which differs from the previous case in notation, and allows the generalization of the precoding scheme for $p > 3$. This is achieved by means of the *zero propagation algorithm*, presented in Section VI. These cases $p = 2, 3$ allow understanding the achievability proof for the general case. Moreover, simulation results are provided in Section VII, where the the sum-rate is depicted as a function of the SNR for different values of p, and DoF achievability is shown. Finally, conclusions are drawn in Section VIII.

C. Notation

We write vectors in boldface lowercase types ($\mathbf{x}$), and matrices in boldface uppercase types ($\mathbf{X}$). We define $\mathbb{R}, \mathbb{C}$ as the real and complex sets of numbers, respectively, while $(\cdot)^T$, $(\cdot)^H$, and $\otimes$ stand for the transpose, transpose and conjugate, and Kronecker product operators, respectively. Also, we define

$$\mathsf{stack}\Big(\mathbf{A}, \mathbf{B}\Big) = \Big[\mathbf{A}^T \;\; \mathbf{B}^T\Big]^T. \tag{1}$$

Furthermore, for any given N-column vector $\mathbf{x} = [x(1), x(2), \ldots, x(N)]^T$ and M-column matrix $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M]$, we define

$$\mathbf{x}(a:b) = [x(a), x(a+1), \ldots, x(b)]^T,$$
$$\mathbf{Y}_{a:b} = [\mathbf{y}_a, \mathbf{y}_{a+1}, \ldots, \mathbf{y}_b].$$

Additionally, $\lceil . \rceil$, $\lfloor . \rfloor$, and $\langle . \rangle$ stand for the ceiling, floor, and modulo-3 operators, respectively. We remark that all indices in this work are assumed to be in the set $\{1, 2, 3\}$, applying the modulo-3 operation only if necessary. Finally, $\mathsf{span}\big(\mathbf{A}\big)$ defines the subspace generated by all linear combinations of the columns of $\mathbf{A}$, and $\mathsf{rank}\,(\mathbf{A})$ denotes its dimension.

II. System Model

The 3-user $(p, p+1)$ MIMO IC is considered, where each transmitter and each receiver is

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equipped with p and $p+1$ antennas, respectively. Each transmitter aims to deliver a message to one unique receiver, labeled with the same index. Perfect and instantaneous channel state information is assumed and exploited at both sides. Channel coefficients are randomly drawn from some continuous complex probability density function, and assumed to be constant along the whole transmission time. The transmission is carried out over $2T$ equivalent channel uses thanks to the T *symbol extensions in time* and *asymmetric complex signaling* [10]. The received and processed signals may be written as follows

$$\mathbf{y}_j = \mathbf{H}_{j,j}\mathbf{V}_j\mathbf{x}_j + \sum_{i=1,i\neq j}^{3} \mathbf{H}_{j,i}\mathbf{V}_i\mathbf{x}_i + \mathbf{n}_j, \qquad (2)$$

$$\mathbf{z}_j = \mathbf{W}_j\mathbf{y}_j, \qquad (3)$$

where $\mathbf{y}_j \in \mathbb{R}^{2T(p+1)\times 1}$ is the received signal vector at the jth receiver, $\mathbf{z}_j \in \mathbb{R}^{\hat{d}_j\times 1}$ is the processed signal, $\mathbf{x}_j \in \mathbb{R}^{\hat{d}_j\times 1}$ is the vector composed of $\hat{d}_j$ real-valued uncorrelated data symbols defining the message intended to the jth receiver, $\mathbf{V}_j \in \mathbb{R}^{2Tp\times\hat{d}_j}$ is the precoding matrix of the jth transmitter, $\mathbf{W}_j \in \mathbb{R}^{\hat{d}_j\times 2T(p+1)}$ is the jth linear receiving filter, and $\mathbf{n}_j \in \mathbb{R}^{2T(p+1)\times 1}$ denotes the noise vector at the jth receiver, whose components are i.i.d. as $\mathcal{N}(0,1)$. Furthermore, $\mathbf{H}_{j,i} \in \mathbb{R}^{2T(p+1)\times 2Tp}$ stands for the equivalent channel matrix from the ith transmitter to the jth receiver after considering symbol extensions in time and asymmetric complex signaling concepts, and applying a change of basis operation, as detailed next.

Let $\bar{\mathbf{H}}_{j,i} \in \mathbb{C}^{(p+1)\times p}$ be the original spatial channel matrix from the ith transmitter to the jth receiver, and assume a transmission over T channel uses. In such a case, one could stack all the received signals, and write a compact system model. The equivalent channel matrix could be written as follows:

$$\mathsf{T}\left(\bar{\mathbf{H}}_{j,i}\right) = \mathbf{I}_T \otimes \bar{\mathbf{H}}_{j,i}, \qquad (4)$$

where $\mathbf{I}_T \in \mathbb{R}^{T\times T}$ is the identity matrix.

Real and imaginary parts of the received signals could be considered separately, as done in [10], and applied to each whole channel matrix. In contrast, here we use the *asymmetric complex signaling* (ACS) concept for each particular channel coefficient. The extended form for each channel element is therefore written as:

$$\mathsf{ACS}\left(\bar{h}_{j,i}^{q,r}\right) = \left|\bar{h}_{j,i}^{q,r}\right| \bar{\mathbf{U}}\left(\bar{\phi}_{j,i}^{q,r}\right) \in \mathbb{R}^{2\times 2}, \qquad (5)$$

where $\bar{h}_{j,i}^{q,r}$ is the *complex* channel gain between the rth antenna of transmitter i and the qth antenna of receiver j, $\bar{\phi}_{j,i}^{q,r}$ is the phase of the complex number $\bar{h}_{j,i}^{q,r}$, and $\bar{\mathbf{U}}\left(\bar{\phi}_{j,i}^{q,r}\right) \in \mathbb{R}^{2\times 2}$ is an unitary matrix given by:

$$\bar{\mathbf{U}}\left(\bar{\phi}_{j,i}^{q,r}\right) = \begin{bmatrix} \cos\left(\bar{\phi}_{j,i}^{q,r}\right) & -\sin\left(\bar{\phi}_{j,i}^{q,r}\right) \\ \sin\left(\bar{\phi}_{j,i}^{q,r}\right) & \cos\left(\bar{\phi}_{j,i}^{q,r}\right) \end{bmatrix}, \qquad (6)$$

with some interesting properties, for example:

$$\bar{\mathbf{U}}(a)\,\bar{\mathbf{U}}(b) = \bar{\mathbf{U}}(a+b)\,,\;\; \bar{\mathbf{U}}(a)^{-1} = \bar{\mathbf{U}}(-a)\,, \qquad (7)$$

for any arbitrary phases $a, b \in [\,0, 2\pi\,]$.

For the sake of clarity, let us write the equivalent channel channel matrix $\hat{\mathbf{H}}_{j,i} \in \mathbb{R}^{2T(p+1)\times 2Tp}$ when the two previous concepts are together applied, given by

$$\hat{\mathbf{H}}_{j,i} = \begin{bmatrix} \mathbf{C}\left(\bar{h}_{j,i}^{1,1}\right) & \dots & \mathbf{C}\left(\bar{h}_{j,i}^{1,p}\right) \\ \vdots & \ddots & \vdots \\ \mathbf{C}\left(\bar{h}_{j,i}^{p+1,1}\right) & \dots & \mathbf{C}\left(\bar{h}_{j,i}^{p+1,p}\right) \end{bmatrix}, \qquad (8)$$

with $\mathbf{C}\left(\bar{h}_{j,i}^{q,r}\right) = \left|\bar{h}_{j,i}^{q,r}\right| \mathbf{I}_T \otimes \bar{\mathbf{U}}\left(\bar{\phi}_{j,i}^{q,r}\right)$. Now the last step to obtain the system model in (2) consists on applying a change of basis (CoB) operation [4]:

$$\mathbf{H}_{j,i} = \mathsf{CoB}\big(\hat{\mathbf{H}}_{j,i}\big) = \mathbf{R}_j\hat{\mathbf{H}}_{j,i}\mathbf{T}_i, \qquad (9)$$

where $\mathbf{R}_j \in \mathbb{R}^{2T(p+1)\times 2T(p+1)}$ and $\mathbf{T}_j \in \mathbb{R}^{2Tp\times 2Tp}$ are invertible linear transformations applied at the transmitters and the receivers. This way the resulting equivalent channel becomes a rotation of $\hat{\mathbf{H}}_{j,i}$ that contains zeros at some specific antenna elements, see [4] for details.

In this work, the same CoB as in [4] is applied at the transmit side, whereas on the receiver side some additional operations described in Appendix A are applied. This way we obtain a simplified structure for the channel matrices, which helps in the precoding design based on interference alignment, as well as the achievability proof.

Remark: Notice that matrices $\mathbf{R}_j$ and $\mathbf{T}_j$ are applied at the transmitter and the receiver, respectively. Therefore, the final precoding matrices at each transmitter and receiver are $\mathbf{W}_j\mathbf{R}_j$ and $\mathbf{T}_j\,\mathbf{V}_j$, respectively.

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III. Degrees of Freedom

The DoF per user d_j of the 3-user $(p, p+1)$ MIMO IC are upper bounded [4] by

$$d_j \leq \frac{p\,(p+1)}{2p+1}. \tag{10}$$

This value was attained using lattice alignment based schemes [14], but it is not known if it coincides with the *linear DoF*. On the other hand, the DoF achieved by the jth user assuming the channel model described in Section II are given by

$$\frac{1}{2T}\mathsf{rank}\left(\mathbf{W}_j\,\mathbf{H}_{j,j}\,\mathbf{V}_j\right) \overset{(a)}{\leq} \frac{\hat{d}_j}{2T} \leq d_j \tag{11}$$

in case all the received interference is completely removed, i.e.

$$\mathbf{W}_j\mathbf{H}_{j,i}\mathbf{V}_i = \mathbf{0}, \quad \forall i \neq j. \tag{12}$$

The previous condition forces $\mathbf{W}_j$ to be an orthogonal projection onto the interference space. Consequently, inequality a in (11) will be satisfied with equality only in case the desired and interference signals are *linearly independent*. Let define the *signal space matrix* (SSM) as the matrix whose columns generate the sum space of desired and interference subspaces at each receiver,

$$\begin{aligned}\mathbf{G}_j &= \left[\,\mathbf{G}_j^{\text{des}}\ \mathbf{G}_j^{\text{int}}\,\right],\\ \mathsf{span}\left(\,\mathbf{G}_j^{\text{des}}\,\right) &= \mathsf{span}\left(\mathbf{H}_{j,j}\mathbf{V}_j\right),\\ \mathsf{span}\left(\,\mathbf{G}_j^{\text{int}}\,\right) &= \mathsf{span}\left(\,\left[\mathbf{H}_{j,j-1}\mathbf{V}_{j-1}\quad \mathbf{H}_{j,j+1}\mathbf{V}_{j+1}\right]\,\right),\end{aligned} \tag{13}$$

where $\mathbf{G}_j^{\text{des}}$ and $\mathbf{G}_j^{\text{int}}$ are some full-rank matrices whose columns form a basis for the desired and interference subspaces, respectively. Given this formulation, proving the DoF achievability reduces to prove that the SSM is full-rank, since in such a case desired and interference signals are linearly independent. Therefore, there exists a set of transmitting and receiving filters simultaneously satisfying inequality a in (11) with equality and (12), i.e. all the desired symbols can be interference-free decoded.

This work proposes a linear precoding/decoding scheme delivering $\hat{d}_j = 2p\,(p+1)$ data symbols to each user, employing ACS and $T = 2p+1$ symbol extensions in time. Then, it is formally proved that the achieved DoF coincide with the DoF outer bound in (10), thus the linear DoF of the channel are characterized.

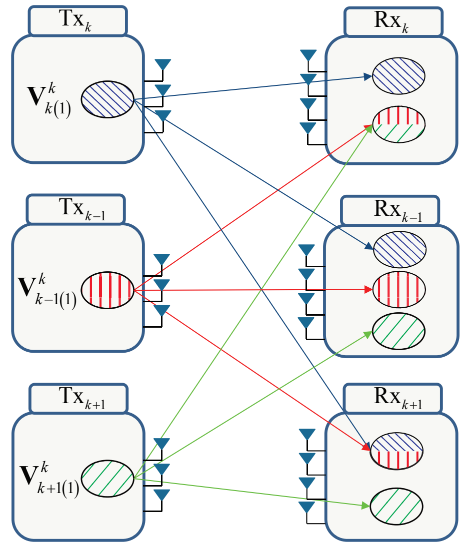

Fig. 2. Occupation of receivers for the signals designed using alignment chain k ($p = 3$ case). Ovals represent different subspaces at transmitters and receivers. Colors/Line patterns identify users.

IV. Precoding Matrix Structure

The *subspace alignment chains* concept [4] describes a linear precoding/decoding strategy whereby the transmit precoders of the different users are connected with the purpose of aligning the interference signals at each receiver. To this end, the precoding matrix of each user is divided in p sub-block matrices, grouped in three main matrix blocks,

$$\mathbf{V}_i = \left[\mathbf{V}^1_{i,(1)}\ \cdots\ \mathbf{V}^1_{i,(S_i^1)}\,\middle|\,\mathbf{V}^2_{i,(1)}\ \cdots\,\middle|\,\cdots\ \mathbf{V}^3_{i,(S_i^3)}\right]\mathbf{P}_i, \tag{14}$$

where $\mathbf{P}_i \in \mathbb{R}^{\hat{d}_j \times \hat{d}_j}$ is a permutation matrix introduced with the purpose of simplifying the description, as detailed in the next sections, and $\mathbf{V}^k_{i,(s)} \in \mathbb{R}^{2Tp \times 2(p+1)}$ denotes the sth sub-block of the ith user designed by means of the kth alignment chain, as explained later.

For any p, three alignment chains are built, describing the constraints to be satisfied by each sub-block, see (16)-(19), where $k = 1, 2, 3$ identifies each chain, $\eta_k = k - p$ is the kth chain last receiver, and the value S_i^k denotes the number of sub-blocks corresponding to the ith user designed according to the kth alignment chain. Since there are 3 users and $3p$ sub-blocks, it may be expressed in closed form as

$$S_i^k = \left\lceil \frac{p - \langle k - i \rangle}{3} \right\rceil. \tag{15}$$

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$$\mathsf{span}\big(\mathbf{H}_{k+1,k}\,\mathbf{V}^k_{k,(1)}\big) = \mathsf{span}\big(\mathbf{H}_{k+1,k-1}\,\mathbf{V}^k_{k-1,(1)}\big) \tag{16}$$

$$\mathsf{span}\big(\mathbf{H}_{k,k-1}\,\mathbf{V}^k_{k-1,(1)}\big) = \mathsf{span}\big(\mathbf{H}_{k,k+1}\,\mathbf{V}^k_{k+1,(1)}\big) \tag{17}$$

$$\mathsf{span}\big(\mathbf{H}_{k-1,k+1}\,\mathbf{V}^k_{k+1,(1)}\big) = \mathsf{span}\big(\mathbf{H}_{k-1,k}\,\mathbf{V}^k_{k,(2)}\big) \tag{18}$$

$$\vdots$$

$$\mathsf{span}\Big(\mathbf{H}_{\eta_k,\eta_k-1}\,\mathbf{V}^k_{\eta_k-1,\left(S^k_{\eta_k-1}\right)}\Big) = \mathsf{span}\Big(\mathbf{H}_{\eta_k,\eta_k+1}\,\mathbf{V}^k_{\eta_k+1,\left(S^k_{\eta_k+1}\right)}\Big) \tag{19}$$

$$\begin{bmatrix} \mathbf{H}_{k+1,k} & -\mathbf{H}_{k+1,k-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k,k-1} & -\mathbf{H}_{k,k-2} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{\eta_k,\eta_k-1} & -\mathbf{H}_{\eta_k,\eta_k+1} \end{bmatrix} \begin{bmatrix} \mathbf{V}^k_{k,(1)} \\ \mathbf{V}^k_{k-1,(1)} \\ \mathbf{V}^k_{k-2,(1)} \\ \mathbf{V}^k_{k,(2)} \\ \vdots \\ \mathbf{V}^k_{\eta_k+1,\left(S^k_{\eta_k+1}\right)} \end{bmatrix} = \mathbf{0} \tag{20}$$

The meaning of (16)-(19) is reviewed in the sequel, and depicted for $p = 3$ in Fig. 2, where ovals represent the subspaces for the kth alignment chain at each transmitter/receiver, and each color/line pattern identifies each user's signals. First, (16) states that the subspace occupied by the sub-block $\mathbf{V}^k_{k,(1)}$ should be the same as that for the sub-block $\mathbf{V}^k_{k-1,(1)}$ at the $(k+1)$th receiver, see Fig. 2. In the literature, this is usually expressed as the *alignment* between sub-block $\mathbf{V}^k_{k,(1)}$ and sub-block $\mathbf{V}^k_{k-1,(1)}$ at receiver $(k+1)$. Next, (17) ensures that this latter sub-block is, *at the same time*, aligned with $\mathbf{V}^k_{k+1,(1)}$ at the kth receiver. This process continues as long as there exists a subspace at each receiver where signals can be aligned. The existence of such subspace can be guaranteed by means of basic linear algebra properties (see [4] for details), and defines the length of the alignment chain, corresponding to the number of sub-blocks designed according to such chain. Notice that the first and last sub-blocks in each alignment chain participate only in the first and the last conditions, respectively. Consequently, they are only aligned with other undesired signals at one of the non-intended receivers. This can be observed in Fig. 2 at receiver $k-1$.

Equations (16)-(19) are usually tackled by dropping the $\mathsf{span}(\,\cdot\,)$ operators, as in (20). Hence, the precoding matrices are obtained as the right null space of some matrix[3]. Notice that (20) represents a sufficient but not necessary condition for (16)-(19). In other words, (20) is more restrictive than (16)-(19), but it is sufficient for our purpose. Finally, for convenience in the analysis each $\mathbf{V}^k_{i,(s)}$ is subsequently divided in p blocks by rows, as follows:

$$\mathbf{V}^k_{i,(s)} = \mathsf{stack}\Big(\mathbf{V}^{k,1}_{i,(s)}, \mathbf{V}^{k,2}_{i,(s)}, \ldots, \mathbf{V}^{k,p}_{i,(s)}\Big), \tag{21}$$

where each $\mathbf{V}^{k,r}_{i,(s)} \in \mathbb{R}^{2T\times 2(p+1)}$ corresponds to one of the $r = 1 \ldots p$ transmit antennas.

[3]For $p = 2$ the notation has to be minorly changed. This case will be addressed in Section V.

V. The $(2,3)$ Case

This section characterizes the linear DoF of the $(2,3)$ constant MIMO IC. The proposed precoding scheme allows each transmitter to deliver $\hat{d}_j = 12$ real-valued symbols to its intended receiver along $2T = 10$ channel extensions, thus attaining the DoF outer bound of $6/5$ in (10). First, the precoding matrices are obtained for this antenna deployment in Section V-A, designed according to minorly modified conditions from the ones shown in (20). Next, Section V-B derives the SSM $\mathbf{G}_j$ introduced in (13) and provides the achievability proof for the proposed precoding scheme.

A. Precoding matrix design

According to definitions (14) and (21), each precoding matrix can be written as

$$\mathbf{V}_i = \begin{bmatrix} \mathbf{V}^1_i & \mathbf{V}^2_i \end{bmatrix}, \quad \mathbf{V}^k_i = \begin{bmatrix} \mathbf{V}^{k,1}_i \\ \mathbf{V}^{k,2}_i \end{bmatrix}, \tag{22}$$

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with $\mathbf{V}_i \in \mathbb{R}^{20\times 12}$, $\mathbf{V}_i^k \in \mathbb{R}^{20\times 6}$ and $\mathbf{V}_i^{k,q} \in \mathbb{R}^{10\times 6}$. Notice that for ease of notation the second subindex (s) appearing in (14) has been dropped.

From (20), the three alignment chains follow. Let us focus on the first alignment chain, given by

$$\left[\, \mathbf{H}_{2,1}, -\mathbf{H}_{2,3} \,\right] \begin{bmatrix} \mathbf{V}_1^1 \\ \mathbf{V}_3^1 \end{bmatrix} = \mathbf{0}\,.$$

By plugging the particular structure of equivalent channels (see Appendix A), it reduces to

$$\begin{bmatrix} \mathbf{C}\left(h_{2,1}^{1,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(h_{2,1}^{2,2}\right) & \mathbf{C}\left(h_{2,3}^{2,1}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{2,3}^{3,2}\right) \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^{1,1} \\ \mathbf{V}_1^{1,2} \\ \mathbf{V}_3^{1,1} \\ \mathbf{V}_3^{1,2} \end{bmatrix} = \mathbf{0}\,.$$

Using properties in (7) and taking into account that non-zero blocks are full-rank with probability one, since channels are drawn from a continuous distribution, one obtains

$$\mathbf{V}_1^{1,1} = \mathbf{0}, \quad \mathbf{V}_3^{1,1} = \mathbf{C}\left(\frac{h_{2,1}^{2,2}}{h_{2,3}^{2,1}}\right) \mathbf{V}_1^{1,2}, \quad \mathbf{V}_3^{1,2} = \mathbf{0}.$$

The other alignment chains are similarly solved, thus at the end we have

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{C}\left(\frac{h_{3,2}^{2,2}}{h_{3,1}^{2,1}}\right) \mathbf{V}_2^{1,2} \\ \mathbf{V}_1^{1,2} & \mathbf{0} \end{bmatrix} \mathbf{P}_1,$$

$$\mathbf{V}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{C}\left(\frac{h_{1,3}^{2,2}}{h_{12}^{2,1}}\right) \mathbf{V}_3^{2,2} \\ \mathbf{V}_2^{1,2} & \mathbf{0} \end{bmatrix} \mathbf{P}_2,$$

$$\mathbf{V}_3 = \begin{bmatrix} \mathbf{C}\left(\frac{h_{2,1}^{2,2}}{h_{2,3}^{2,1}}\right) \mathbf{V}_1^{1,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_3^{2,2} \end{bmatrix} \mathbf{P}_3.$$

Now we will make use of the permutation matrices $\mathbf{P}_i$ in order to obtain the same structure for all precoding matrices. Notice that reordering the columns of the precoders does not affect to the interference alignment. Furthermore, notice that there are only three non-zero precoding sub-blocks. Hereafter, they will be referred to as the *support precoding blocks* (SPBs) and denoted as $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$. Therefore, the jth precoding matrix for $j = 1, 2, 3$ is generally written as follows:

$$\mathbf{V}_j = \begin{bmatrix} \mathbf{C}\left(\frac{h_{j-1,j+1}^{2,2}}{h_{j-1,j}^{2,1}}\right) \mathbf{A}_{j+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_j \end{bmatrix}. \qquad (23)$$

B. Achievability proof

This section derives the SSM $\mathbf{G}_j$ as a function of the SPBs. Then, a design for those matrices is proposed easing the achievability proof, formalized in Lemma 1.

For the proper computation of the SSM, let write

$$\mathbf{H}_{j,j+1}\mathbf{V}_{j+1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(h_{j,j-1}^{2,2}\right)\mathbf{A}_{j-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(h_{j,j+1}^{3,2}\right)\mathbf{A}_{j+1} \end{bmatrix} \qquad (24)$$

$$\mathbf{H}_{j,j-1}\,\mathbf{V}_{j-1} = \begin{bmatrix} \mathbf{C}\left(\frac{h_{j,j-1}^{1,1} h_{j+1,j}^{2,2}}{h_{j+1,j-1}^{2,1}}\right)\mathbf{A}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(h_{j,j-1}^{2,2}\right)\mathbf{A}_{j-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad (25)$$

defining the subspaces of received interference at the jth receiver, see (13). Notice that the first block column of (24) is aligned with the last block column of (25), which is actually forced by the alignment chain $j+1$. As a result, the basis for the interference space $\mathbf{G}_j^{\text{int}}$ is defined by the three linearly independent block columns of (24)-(25), and the SSM $\mathbf{G}_j$ is given by (27).

The SSM obtained in (27) is similar to the equivalent magnitude obtained in equation (16) of [21]. Even though in this case the full-rank condition for the SSM can be ensured by picking entries of the SPBs randomly (as pointed out by [21]), we present a formal proof that is also useful for the $p > 2$ case.

Let define $\boldsymbol{\lambda}_i^j \in \mathbb{R}^{6\times 1}$, $i = 1 \ldots 5$, $j = 1, 2, 3$ as the *rank descriptors*. Then, one may ensure that the SSM is full-rank iff the only solution for

$$\mathbf{G}_j \left[\, \left(\boldsymbol{\lambda}_1^j\right)^T \ldots \left(\boldsymbol{\lambda}_5^j\right)^T \,\right]^T = \mathbf{0} \qquad (26)$$

is to set all rank descriptors to zero. To this end, let also define an arbitrary orthonormal basis $\mathbf{B} = \left[\mathbf{b}_1\, \mathbf{b}_2\, \ldots\, \mathbf{b}_{10}\right] \in \mathbb{R}^{10\times 10}$. We propose the fol-

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$$\mathbf{G}_j = \begin{bmatrix} \mathbf{C}\left(h_{j,j}^{1,1}\right)\mathbf{A}_{j+1} & \mathbf{C}\left(h_{j,j}^{1,2}\right)\mathbf{A}_j & \mathbf{C}\left(\frac{h_{j,j-1}^{1,1}h_{j+1,j}^{2,2}}{h_{j+1,j-1}^{2,1}}\right)\mathbf{A}_j & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(h_{j,j}^{2,1}\right)\mathbf{A}_{j+1} & \mathbf{C}\left(h_{j,j}^{2,2}\right)\mathbf{A}_j & \mathbf{0} & \mathbf{C}\left(h_{j,j-1}^{2,2}\right)\mathbf{A}_{j-1} & \mathbf{0} \\ \mathbf{C}\left(h_{j,j}^{3,1}\right)\mathbf{A}_{j+1} & \mathbf{C}\left(h_{j,j}^{3,2}\right)\mathbf{A}_j & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{j,j+1}^{3,2}\right)\mathbf{A}_{j+1} \end{bmatrix} \quad (27)$$

lowing design:

$$\begin{aligned} \mathbf{A}_1 &= \left[\mathbf{B}_{1:2} \quad \mathbf{B}_{3:5} \quad \mathbf{B}_6\right], \\ \mathbf{A}_2 &= \left[\mathbf{B}_{1:2} \quad \mathbf{B}_{7:9} \quad \mathbf{B}_{10}\right], \\ \mathbf{A}_3 &= \left[\mathbf{B}_{3:5} \quad \mathbf{B}_{7:9}\right]. \end{aligned} \quad (28)$$

The following lemma states the DoF achievability:

Lemma 1 ($\mathbf{G}_j$ full-rank for $p = 2$): Considering (27) and the SPBs chosen as in (28), then the only possible solution for (26) is $\boldsymbol{\lambda}_i^j = \mathbf{0}, \forall i, j$.

Proof: See Appendix B. □

Finally, the optimal DoF are settled as follows:

Theorem 1 (DoF for the (2,3) case): The 3-user $(2, 3)$ MIMO IC with constant channel coefficients has exactly 6/5 linear DoF per user.

Proof: Each user transmits $\hat{d} = 12$ real-valued symbol streams along $T = 5$ symbol extensions in time, considering ACS, and the precoding scheme described in Section V-A. Therefore, according to Lemma 1, the SSM $\mathbf{G}_j$ is full rank, thus interference and desired signals become linearly independent, and the desired symbols can be decoded. Since the DoF outer bound in (10) and the achievable DoF attained by the proposed scheme match, this value corresponds to the optimal linear DoF. □

VI. The $(p, p+1)$ case with $p > 2$

This section address the $(p, p+1)$ MIMO IC with constant channel coefficients and $p \geq 3$. The proposed proposed precoding scheme allows each user to obtain $\hat{d}_j = 2p(p+1)$ real-valued data symbols over $2T = 2(2p+1)$ channel extensions, thus attaining the DoF outer bound of $\frac{p(p+1)}{2p+1}$ in (10).

Unfortunately, the number of conditions used for the precoder design, see (20), increases with p^2. Therefore, the analysis using the approach for the $p = 2$ case gets complicated as p grows. This section presents a methodology to simplify the resolution of such matrix equation system, which will be illustrated for the $p = 3$ case. The core of this methodology is the zero propagation algorithm, which allows to obtain the structure of the transmit and receive filters for any value of p.

A. Precoding matrix design

Out of the three alignment chains, let us consider the first alignment chain ($k = 1$) given by (31), shown at the top of the next page.The remaining alignment chains are handled similarly.

It can be observed that thanks to the obtained structure of matrix $\mathbf{E}$, some sub-blocks of $\mathbf{F}$ are zero. For example, consider the fifth block row element of $\mathbf{E} \cdot \mathbf{F}$:

$$\mathbf{C}\left(h_{1,3}^{1,1}\right)\mathbf{V}_{3,(1)}^{1,1} = \mathbf{0}. \quad (29)$$

Clearly, the only solution above is $\mathbf{V}_{3,(1)}^{1,1} = \mathbf{0}$ with probability one. Hence, other equations where this variable participates are simplified. Each of these events is denoted as a *zero propagation* (ZP) and give the possibility of finding which blocks are zero for $\mathbf{F}$ in (31). Inspired by this idea, we present the ZP algorithm, see Table I. This algorithm allows simplifying the conditions initially presented in (31), reducing the number of blocks to be designed in (32). Moreover, by writing the remaining equations it turns out that each precoding matrix can be written as a function of three SPBs, as follows:

$$\mathbf{V}_i = \begin{bmatrix} \mathbf{C}\left(\theta_{i,(1)}^{i-1,1}\right)\mathbf{A}_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(\theta_{i,(1)}^{i-1,2}\right)\mathbf{A}_{i-1} & \mathbf{C}\left(\theta_{i,(1)}^{i+1,2}\right)\mathbf{A}_{i+1} & \mathbf{C}\left(\theta_{i,(1)}^{i,2}\right)\mathbf{A}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_i \end{bmatrix}, \quad (30)$$

where $\theta_{i,(1)}^{q,r}$ stands for the complex value obtained from the qth alignment chain and located at the rth block row of $\mathbf{V}_i$. They can be obtained by computing a null space basis from (32). Note that the number of unknown sub-block matrices is reduced from 27 in (31) to 3 in (30). In general, the p^2 variables (block matrices) involved in each of the three alignment chains can be written as a function

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$$\begin{bmatrix} \mathbf{C}\left(h_{2,1}^{1,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(h_{2,1}^{2,2}\right) & \mathbf{C}\left(h_{2,1}^{2,3}\right) & \mathbf{C}\left(h_{2,3}^{2,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{2,1}^{3,3}\right) & \mathbf{C}\left(h_{2,3}^{3,1}\right) & \mathbf{C}\left(h_{2,3}^{3,2}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{2,3}^{4,3}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,3}^{1,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,3}^{2,2}\right) & \mathbf{C}\left(h_{1,3}^{2,3}\right) & \mathbf{C}\left(h_{1,2}^{2,1}\right) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,3}^{3,3}\right) & \mathbf{C}\left(h_{1,2}^{3,1}\right) & \mathbf{C}\left(h_{1,2}^{3,2}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,2}^{4,3}\right) \end{bmatrix} \begin{bmatrix} \mathbf{V}_{1,(1)}^{1,1} \\ \mathbf{V}_{1,(1)}^{1,2} \\ \mathbf{V}_{1,(1)}^{1,3} \\ \mathbf{V}_{3,(1)}^{1,1} \\ \mathbf{V}_{3,(1)}^{1,2} \\ \mathbf{V}_{3,(1)}^{1,3} \\ \mathbf{V}_{2,(1)}^{1,1} \\ \mathbf{V}_{2,(1)}^{1,2} \\ \mathbf{V}_{2,(1)}^{1,3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{E} \cdot \mathbf{F} = \mathbf{0} \tag{31}$$

$$\begin{bmatrix} \mathbf{C}\left(h_{2,1}^{2,2}\right) & \mathbf{C}\left(h_{2,1}^{2,3}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(h_{2,1}^{3,3}\right) & \mathbf{C}\left(h_{2,3}^{3,2}\right) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,3}^{2,2}\right) & \mathbf{C}\left(h_{1,2}^{2,1}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{1,2}^{3,1}\right) & \mathbf{C}\left(h_{1,2}^{3,2}\right) \end{bmatrix} \begin{bmatrix} \mathbf{V}_{1,(1)}^{1,2} \\ \mathbf{V}_{1,(1)}^{1,3} \\ \mathbf{V}_{3,(1)}^{1,2} \\ \mathbf{V}_{2,(1)}^{1,1} \\ \mathbf{V}_{2,(1)}^{1,2} \end{bmatrix} = \mathbf{0} \tag{32}$$

TABLE I
ZERO PROPAGATION ALGORITHM

Consider the matrix equation system given by $\mathbf{E} \cdot \mathbf{F} = \mathbf{0}$, with $\mathbf{F} \in \mathbb{R}^{F_{\text{BR}} \cdot r_{\text{F}} \times F_{\text{BC}}}$ and $\mathbf{E} \in \mathbb{R}^{E_{\text{BR}} \cdot r_{\text{E}} \times F_{\text{BR}} \cdot r_{\text{F}}}$, where $r_{\text{F}}(r_{\text{E}})$ defines the number of block rows of $\mathbf{F}$ ($\mathbf{E}$). Moreover, F_{BC} (F_{BR}) defines the number of columns (rows) of each block element of $\mathbf{F}$, and E_{BR} defines the number of rows of each block column of $\mathbf{E}$. The blocks of $\mathbf{F}$ that can be set to zero may be obtained by computing the following steps:

1. Find one block row in $\mathbf{E}$ containing only one non-zero element, located at the $[r^*, c^*]$th block position.
2. Set $\begin{cases} \mathbf{E}(r^*, :) = \text{zeros}(E_{\text{BR}}, F_{\text{BR}} \cdot r_{\text{F}}) \\ \mathbf{E}(:, c^*) = \text{zeros}(E_{\text{BR}} \cdot r_{\text{E}}, F_{\text{BR}}) \end{cases}$
3. Set $\mathbf{F}(c^*, :) = \text{zeros}(F_{\text{BR}}, F_{\text{BC}})$.
4. Repeat (1)-(3) until (1) provides no more block rows.
5. Remove the all-zeros block columns and block columns in $\mathbf{E}$ and $\mathbf{F}$, respectively.

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of one SPB of dimension $2(2p+1)\times 2(p+1)$. Hence, after applying the ZP algorithm the problem reduces to the design of the three SPBs, one obtained from each alignment chain.

B. Achievability proof

This section derives the SSM for the $p=3$ case, and generalizes the ideas for $p>3$. First, a design for the three SPBs in (30) is proposed, generalizing (28) for any value of p. Second, the SSM is shown to be full rank, thus the optimal linear DoF are stated in Theorem 2.

In order to build $\mathbf{G}_j$, it is necessary to compute a basis for the sum space defined by the received interference and desired signals. Regarding the desired signals, it can be easily seen that $\mathbf{G}_j^{\text{des}} = \mathbf{H}_{j,j}\mathbf{V}_j$. On the other hand, since some of the interference is aligned it is necessary to first calculate the products $\mathbf{H}_{j,j-1}\mathbf{V}_{j-1}$ and $\mathbf{H}_{j,j+1}\mathbf{V}_{j+1}$. Next, we will see that this task can be highly alleviated. Note that the ZP algorithm output in (32) not only states which subblocks of each $\mathbf{V}_i$ are actually zero, but also which conditions should be satisfied by the remaining subblocks. For example, (32) forces

$$\mathbf{C}\left(h_{2,1}^{2,2}\right)\mathbf{V}_{1,(1)}^{1,2} + \mathbf{C}\left(h_{2,1}^{2,3}\right)\mathbf{V}_{1,(1)}^{1,3} = \mathbf{0} \quad (33)$$

Interestingly, this is indeed one of the elements resulting from the product $\mathbf{H}_{2,1}\mathbf{V}_1$. Taking into account all other conditions where there are only elements managed by one unique transmitter, the products $\mathbf{H}_{j,j-1}\mathbf{V}_{j-1}$ and $\mathbf{H}_{j,j+1}\mathbf{V}_{j+1}$ can be further simplified, obtaining (37)-(38), where $\bar{\theta}_{j,i}^{q,r}$ is the corresponding complex number for the (q,r)th position of $\mathbf{H}_{j,i}\mathbf{V}_i, i \neq j$. Note that in this case due to alignment conditions, we will have $\bar{\theta}_{j,j-1}^{q,q} = \bar{\theta}_{j,j+1}^{q,q-1}$ with $q=2,3$, i.e. columns $\{2,3\}$ of $\mathbf{H}_{j,j+1}\mathbf{V}_{j+1}$ are aligned with columns $\{1,2\}$ of $\mathbf{H}_{j,j-1}\mathbf{V}_{j-1}$, respectively. Therefore, in this case the SSM is given by (39)-(40), where $\mathbf{G}_j^{\text{des}}(q,r)$ and $\hat{\theta}_j^{q,r}$ are the matrix and the complex number corresponding to the (q,r)th position of $\mathbf{G}_j^{\text{des}}$ and $\mathbf{G}_j^{\text{int}}$, respectively. For $\mathbf{G}_j^{\text{des}}$, we write the blocks $\mathbf{G}_j^{\text{des}}(q,r)$, because they are linear combinations of some extended channel elements, e.g.

$$\mathbf{G}_j^{\text{des}}(1,2) = \mathbf{C}\left(h_{1,1}^{1,1}\right) - \mathbf{C}\left(\frac{h_{1,1}^{2,2}h_{3,1}^{3,1}}{h_{3,1}^{3,2}}\right).$$

Notice that each matrix $\mathbf{C}\left(\hat{\theta}_j^{q,r}\right)$ in $\mathbf{G}_j^{\text{int}}$ is a combination of a number of cross-channels coefficients, thus it can be assumed independent of any of the matrices $\mathbf{G}_j^{\text{des}}(q,r)$, which are also function of direct channel coefficients.

In contrast to (27), now it is not clear if the SSM for this case is full-rank by just taking the SPBs randomly. Next, we provide the proof to verify that $\mathbf{G}_j$ is full rank. Magnitudes are defined for a general value of p, and all possible procedures are generalized.

As before, the SSM may be shown to be full rank iff all $\boldsymbol{\lambda}_i^j \in \mathbb{R}^{2(2p+1)\times 1}$, $i=1\ldots 2p+1$, $j=1,2,3$ constrained by

$$\mathbf{G}_j \left[\left(\boldsymbol{\lambda}_1^j\right)^T \ldots \left(\boldsymbol{\lambda}_{2p+1}^j\right)^T \right]^T = \mathbf{0} \quad (34)$$

are actually equal to zero. Define an orthonormal basis $\mathbf{B} = \left[\mathbf{b}_1, \mathbf{b}_2 \ldots \mathbf{b}_{2(2p+1)}\right] \in \mathbb{R}^{2(2p+1)\times 2(2p+1)}$ and

$$\begin{aligned} X_1 &= \{3,4,\ldots,p+3\}, \\ X_2 &= \{p+4,p+5,\ldots,2p+2\}, \\ Y_1 &= \{2p+3,\ldots,3p+3\}, \\ Y_2 &= \{3p+4,\ldots,4p+2\}, \\ Z &= \{1,2\}. \end{aligned} \quad (35)$$

Hereafter, we use these sets to arrange columns of $\mathbf{B}$, e.g. $\mathbf{B}_{X_2} = \mathbf{B}_{p+4:2p+2}$. Accordingly, we set:

$$\begin{aligned} \mathbf{A}_1 &= \left[\mathbf{B}_Z \quad \mathbf{B}_{X_1} \quad \mathbf{B}_{X_2}\right], \\ \mathbf{A}_2 &= \left[\mathbf{B}_Z \quad \mathbf{B}_{Y_1} \quad \mathbf{B}_{Y_2}\right], \\ \mathbf{A}_3 &= \left[\mathbf{B}_{X_1} \quad \mathbf{B}_{Y_1}\right]. \end{aligned} \quad (36)$$

Given these definitions, the following lemma states the DoF achievability:

Lemma 2 ($\mathbf{G}_j$ full-rank for $p=3\ldots 6$): For the $3 \le p \le 6$ cases, the SSM defined as in (13) with SPBs chosen as in (36) is full rank with probability one.

Proof: See Appendix C. □

Finally, the DoF characterization for $p=3\ldots 6$ follows from Lemma 2, and it is next formalized:

Theorem 2 (DoF of the $(p,p+1)$ IC, $p=3\ldots 6$): The 3-user $(p,p+1)$, $3 \le p \le 6$ MIMO IC with constant channel coefficients has exactly $\frac{p(p+1)}{2p+1}$ linear DoF per user.

Proof: The proof is analogous to the proof

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$$\mathbf{H}_{j,j-1}\mathbf{V}_{j-1} = \begin{bmatrix} \mathbf{C}\left(\bar{\theta}_{j,j-1}^{1,1}\right)\mathbf{A}_{j+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(\bar{\theta}_{j,j-1}^{2,1}\right)\mathbf{A}_{j+1} & \mathbf{C}\left(\bar{\theta}_{j,j-1}^{2,2}\right)\mathbf{A}_{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(\bar{\theta}_{j,j-1}^{3,3}\right)\mathbf{A}_{j-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{37}$$

$$\mathbf{H}_{j,j+1}\mathbf{V}_{j+1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(\bar{\theta}_{j,j+1}^{2,1}\right)\mathbf{A}_{j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\left(\bar{\theta}_{j,j+1}^{3,2}\right)\mathbf{A}_{j-1} & \mathbf{C}\left(\bar{\theta}_{j,j+1}^{3,3}\right)\mathbf{A}_{j+1} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(\bar{\theta}_{j,j+1}^{4,3}\right)\mathbf{A}_{j+1} \end{bmatrix} \tag{38}$$

$$\mathbf{G}_j^{\text{des}} = \begin{bmatrix} \mathbf{G}_j^{\text{des}}(1,1)\,\mathbf{A}_{j-1} & \mathbf{G}_j^{\text{des}}(1,2)\,\mathbf{A}_{j+1} & \mathbf{G}_j^{\text{des}}(1,3)\,\mathbf{A}_{j} \\ \mathbf{G}_j^{\text{des}}(2,1)\,\mathbf{A}_{j-1} & \mathbf{G}_j^{\text{des}}(2,2)\,\mathbf{A}_{j+1} & \mathbf{G}_j^{\text{des}}(2,3)\,\mathbf{A}_{j} \\ \mathbf{G}_j^{\text{des}}(3,1)\,\mathbf{A}_{j-1} & \mathbf{G}_j^{\text{des}}(3,2)\,\mathbf{A}_{j+1} & \mathbf{G}_j^{\text{des}}(3,3)\,\mathbf{A}_{j} \\ \mathbf{G}_j^{\text{des}}(4,1)\,\mathbf{A}_{j-1} & \mathbf{G}_j^{\text{des}}(4,2)\,\mathbf{A}_{j+1} & \mathbf{G}_j^{\text{des}}(4,3)\,\mathbf{A}_{j} \end{bmatrix} \tag{39}$$

$$\mathbf{G}_j^{\text{int}} = \begin{bmatrix} \mathbf{C}\left(\hat{\theta}_j^{1,1}\right)\mathbf{A}_{j+1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\left(\hat{\theta}_j^{2,1}\right)\mathbf{A}_{j+1} & \mathbf{C}\left(\hat{\theta}_j^{2,2}\right)\mathbf{A}_{j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(\hat{\theta}_j^{2,3}\right)\mathbf{A}_{j-1} & \mathbf{C}\left(\hat{\theta}_j^{2,4}\right)\mathbf{A}_{j+1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(\hat{\theta}_j^{3,4}\right)\mathbf{A}_{j+1} \end{bmatrix} \tag{40}$$

of Theorem 1. In general, the optimal linear DoF are achieved by using the proposed transmission scheme, delivering $\hat{d}_j = 2p(p+1)$ symbol streams to each user along $2T = 2(2p+1)$ symbol extensions in time, and considering ACS. □

We would want to remark that we have only analytically proved the cases $p = 2, 3, \ldots, 6$. Nonetheless, based on the explained methodology and some numerical results (see next section), we conjecture that the ZP algorithm provides full rank SSMs for $p > 6$, and hence the optimal DoF can be attained and proved:

Conjecture 1 (DoF for the general $(p, p+1)$ IC): The 3-user $(p, p+1)$ MIMO IC with constant channel coefficients has exactly $\frac{p(p+1)}{2p+1}$ linear DoF per user for $p > 6$, and they can be achieved by means of applying subspace alignments chains, symbol extensions in time and ACS.

VII. NUMERICAL RESULTS

In order to validate the contributions of this work, as well as increase the strength of Conjecture 1, we simulate the cases $p = 2, 3, 5, 6, 8, 9$ for the 3-user MIMO IC. Two schemes are simulated, the one proposed in this work, and the design in [4] not considering ACS. In both cases, we apply the CoB operation and the additional transformations as explained in Appendix A together with the proposed scheme. Results are shown in Fig. 3, where solid/dashed lines denote the two schemes with/without considering ACS. It can be seen that the scheme considering ACS improves the slope achieved at high SNR for each case. Moreover, for the cases $p > 6$ simulated, the slope at high SNR follows our conjecture on the DoF.

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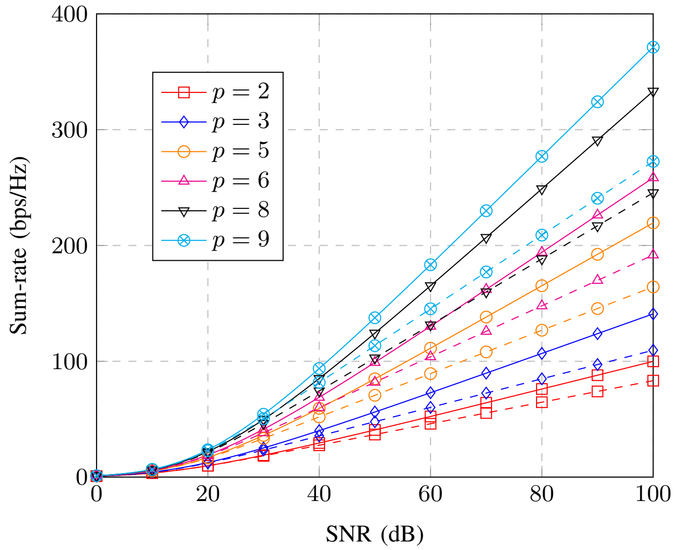

Fig. 3. Comparison of using IA with the proposed channel extension (solid lines) with respect to using only symbol extensions in time (dashed lines).

VIII. Conclusions

This work has investigated the linear DoF of the 3-user $(p, p+1)$ MIMO Interference Channel with constant channel coefficients and full channel state information at both sides. By means of the proposed precoding scheme, the optimal linear DoF achievability has been proved for the cases $p = 2 \ldots 6$. Moreover, a methodology has been presented easing the proof for any value of p, where we conjecture that the optimal linear DoF can also be attained. This conjecture has been numerically checked for two cases.

The contribution of this work is twofold. On the one hand, we have shown that the use of asymmetric complex signaling together with the previous state-of-the-art approach in [4] allows characterizing the linear DoF of this channel. Therefore, we have provided a formal proof, and uncoupled the achievability statement from numerical experiments. On the other hand, our results revealed that, except for the SISO case, the same DoF can be achieved using linear IA with respect to lattice alignment.

Future work is oriented to avoid the need of using lattice alignment schemes also for the SISO case, where linear DoF inner and outer bounds have not yet been found. Also, it may be interesting to optimize not only the slope of the sum-rate curve at the high SNR regime, but also the SNR offset. Further improvement seems to be possible by optimizing the SPBs in terms of the sum-rate subject to some transmit power constraint.

Appendix A
Additional Change of Basis at the Receiver Side

The CoB operation [4] is a tool that provides a predetermined structure for the cross-channel matrices. In particular, it forces zeros at some specific antenna elements. For example, the equivalent cross-channel matrices $\{\tilde{\mathbf{H}}_{j,j-1}, \tilde{\mathbf{H}}_{j,j+1}\}$ for $p = 3$ after performing the original CoB described in [4] are given by (42). Here we assume that the CoB at the receiver $\mathbf{R}_j$ is the product of two matrices: the original CB and an additional combining matrix $\boldsymbol{\Upsilon}_j \in \mathbb{R}^{2T(p+1)\times 2T(p+1)}$ such that (43) is satisfied. Then, each block row of $\boldsymbol{\Upsilon}_j = \left[\boldsymbol{\upsilon}_{j,1}^T, \ldots, \boldsymbol{\upsilon}_{j,4}^T\right]^T$ is derived as follows:

$$\boldsymbol{\upsilon}_{j,1} = \begin{bmatrix} \mathbf{I}_{2T} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{\upsilon}_{j,2} = \mathsf{null}\left(\left[\tilde{\mathbf{H}}_{j,j-1}(\,:,1), \tilde{\mathbf{H}}_{j,j+1}(\,:,2:3)\right]\right),$$
$$\boldsymbol{\upsilon}_{j,3} = \mathsf{null}\left(\left[\tilde{\mathbf{H}}_{j,j-1}(\,:,1:2), \tilde{\mathbf{H}}_{j,j+1}(\,:,3)\right]\right),$$
$$\boldsymbol{\upsilon}_{j,4} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{2T} \end{bmatrix},$$

where $\mathbf{A}(:, b : c)$ gives the matrix resulting from picking the entries of $\mathbf{A}$ from block column b to c, and $\mathbf{I}_{2T} \in \mathbb{R}^{2T\times 2T}$, $\mathbf{0} \in \mathbb{R}^{2T\times 2Tp}$ are the identity and all-zero matrices.

Appendix B
Proof of Lemma 1

We will prove the lemma for the system of equations defined for $j = 1$. The cases $j = 2, 3$ can be similarly handled, and they are omitted to avoid redundancy. Therefore, we drop the supraindex j and write $\boldsymbol{\lambda}_i, i = 1, \ldots 5$ to simplify notation. Additionally, some rank-preserving transformations will be applied to $\mathbf{G}_j$. Consequently, the matrix equation system in (26) for $j = 1$ can be written as follows:

$$\begin{gathered} \mathbf{C}\left(h_{1,1}^{1,1}\right)\mathbf{A}_2\boldsymbol{\lambda}_1 + \mathbf{A}_1\boldsymbol{\lambda}_3 = \mathbf{0}, \\ \mathbf{C}\left(h_{1,1}^{2,1}\right)\mathbf{A}_2\boldsymbol{\lambda}_1 + \mathbf{C}\left(h_{1,1}^{2,2}\right)\mathbf{A}_1\boldsymbol{\lambda}_2 + \mathbf{A}_3\boldsymbol{\lambda}_4 = \mathbf{0}, \\ \mathbf{C}\left(h_{1,1}^{3,2}\right)\mathbf{A}_1\boldsymbol{\lambda}_2 + \mathbf{A}_2\boldsymbol{\lambda}_5 = \mathbf{0}, \end{gathered} \quad (41)$$

which can be simplified by introducing (28), and by means of linear independence among the vectors $\mathbf{b}_i$.

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$$\left[\tilde{\mathbf{H}}_{j,j-1},\tilde{\mathbf{H}}_{j,j+1}\right]=\begin{bmatrix}\mathbf{C}\left(\tilde{h}_{j,j-1}^{1,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{C}\left(\tilde{h}_{j,j-1}^{2,1}\right) & \mathbf{C}\left(\tilde{h}_{j,j-1}^{2,2}\right) & \mathbf{C}\left(\tilde{h}_{j,j-1}^{2,3}\right) & \mathbf{C}\left(\tilde{h}_{j,j+1}^{2,1}\right) & \mathbf{0} & \mathbf{C}\left(\tilde{h}_{j,j+1}^{2,3}\right)\\ \mathbf{C}\left(\tilde{h}_{j,j-1}^{3,1}\right) & \mathbf{0} & \mathbf{C}\left(\tilde{h}_{j,j-1}^{3,3}\right) & \mathbf{C}\left(\tilde{h}_{j,j+1}^{3,1}\right) & \mathbf{C}\left(\tilde{h}_{j,j+1}^{3,2}\right) & \mathbf{C}\left(\tilde{h}_{j,j+1}^{3,3}\right)\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(\tilde{h}_{j,j+1}^{4,3}\right)\end{bmatrix} \quad (42)$$

$$\boldsymbol{\Upsilon}_j\left[\tilde{\mathbf{H}}_{j,j-1},\tilde{\mathbf{H}}_{j,j+1}\right]=\begin{bmatrix}\mathbf{C}\left(h_{j,j-1}^{1,1}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\left(h_{j,j-1}^{2,2}\right) & \mathbf{C}\left(h_{j,j-1}^{2,3}\right) & \mathbf{C}\left(h_{j,j+1}^{2,1}\right) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{j,j-1}^{3,3}\right) & \mathbf{C}\left(h_{j,j+1}^{3,1}\right) & \mathbf{C}\left(h_{j,j+1}^{3,2}\right) & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}\left(h_{j,j+1}^{4,3}\right)\end{bmatrix} \quad (43)$$

For instance, consider all equations corresponding to $\mathbf{B}_{1:2}$ in (41):

$$\mathbf{C}\left(h_{1,1}^{1,1}\right)\mathbf{b}_q\,\lambda_1(q)+\mathbf{b}_q\,\lambda_3(q)=\mathbf{0}, \quad (44)$$

$$\mathbf{C}\left(h_{1,1}^{2,1}\right)\mathbf{b}_q\,\lambda_1(q)+\mathbf{C}\left(h_{1,1}^{2,2}\right)\mathbf{b}_q\,\lambda_2(q)=\mathbf{0}, \quad (45)$$

$$\mathbf{C}\left(h_{1,1}^{3,2}\right)\mathbf{b}_q\,\lambda_2(q)+\mathbf{b}_q\,\lambda_5(q)=\mathbf{0}, \quad (46)$$

with $q=1,2$. Each of such equations can be simplified as follows. Let us define:

$$\tilde{\mathbf{b}}_q=\begin{bmatrix}b_q(1)+jb_q(2)\\ b_q(3)+jb_q(4)\\ \vdots\\ b_q(9)+jb_q(10)\end{bmatrix} \quad (47)$$

where $\mathbf{b}_q=\mathsf{stack}\Big(b_q(1),\ldots,b_q(10)\Big)$, $j=\sqrt{-1}$ stands for the imaginary unit, and $q=1,2$. Then, as in [10], we can write (44)-(46) in terms of $\tilde{\mathbf{b}}_q$. For instance, (44) can be rewritten as follows:

$$\begin{aligned}\left|h_{1,1}^{1,1}\right|e^{j\phi_{1,1}^{1,1}}\tilde{\mathbf{b}}_q\,\lambda_1(q)+\tilde{\mathbf{b}}_q\,\lambda_3(q)&=\mathbf{0},\\ \left|h_{1,1}^{1,1}\right|e^{j\phi_{1,1}^{1,1}}\lambda_1(q)+\lambda_3(q)&=0,\end{aligned} \quad (48)$$

with $q=1,2$. Hence, equating real and imaginary parts of each equation to zero, we have:

$$\begin{aligned}\left|h_{1,1}^{1,1}\right|\sin\left(\phi_{1,1}^{1,1}\right)\lambda_1(q)&=0,\\ \left|h_{1,1}^{1,1}\right|\cos\left(\phi_{1,1}^{1,1}\right)\lambda_1(q)+\lambda_3(q)&=0,\end{aligned} \quad (49)$$

with $q=1,2$. The set containing all the possible values such that $\left|h_{1,1}^{1,1}\right|\sin\left(\phi_{1,1}^{1,1}\right)=0$ is a countable set, thus it has zero measure [22]. By randomness arguments the only solution is $\lambda_r(q)=0, r=1,3, q=1,2$. Applying this methodology to all equations derived from all groups of columns of $\mathbf{B}$, one finds out that all rank descriptors must be zero.

An alternative way to prove that the rank descriptors associated to $\mathbf{B}_{1:2}$ must be zero is next shown. Instead of developing (44) only, consider all equations (44)-(46) in the form of (49). Then, equating imaginary parts to zero, we have

$$\begin{bmatrix}\left|h_{1,1}^{1,1}\right|\sin\left(\phi_{1,1}^{1,1}\right) & 0\\ \left|h_{1,1}^{2,1}\right|\sin\left(\phi_{1,1}^{2,1}\right) & \left|h_{1,1}^{2,2}\right|\sin\left(\phi_{1,1}^{2,2}\right)\\ 0 & \left|h_{1,1}^{3,2}\right|\sin\left(\phi_{1,1}^{3,2}\right)\end{bmatrix}\begin{bmatrix}\lambda_1(q)\\ \lambda_2(q)\end{bmatrix}=\mathbf{0}\,. \quad (50)$$

We will refer to the 3×2 matrix at the left-hand side of (50) as an *elimination matrix*. As long as we can ensure it has no right null space, all rank descriptors in (50) can be set to zero. In this case, this is trivially ensured by means of randomness arguments and the matrix dimensions. Likewise, using the real counterpart of (50), we have $\lambda_i(q)=0, i=3,5$, $q=1,2$. By the same rationale applied to each group of columns of $\mathbf{B}$, we obtain an elimination matrix for each case, and it is easy to check that none of them has right null space, thus all rank descriptors are definitely equal to zero.

So far we have proved that considering ACS is sufficient for achieving a full rank SSM. In what follows, we explain why it is necessary when using the scheme based on alignment chains. In this regard, notice that if only symbol extensions in time are employed, a set of equations similar to (44)-(46) is obtained, and we have

$$\begin{aligned} h_{1,1}^{1,1}\lambda_1(q) + \lambda_3(q) = 0 \\ h_{1,1}^{2,1}\lambda_1(q) + h_{1,1}^{2,2}\lambda_2(q) = 0 \\ h_{1,1}^{3,2}\lambda_2(q) + \lambda_5(q) = 0 \end{aligned} \quad (51)$$

written in matrix form as

$$\begin{bmatrix} h_{1,1}^{1,1} & 0 & 1 & 0 \\ h_{1,1}^{2,1} & h_{1,1}^{2,2} & 0 & 0 \\ 0 & h_{1,1}^{3,2} & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1(q) \\ \lambda_2(q) \\ \lambda_3(q) \\ \lambda_5(q) \end{bmatrix} = \mathbf{0}, \quad (52)$$

with $q = 1, 2$. Note that the matrices $\mathbf{C}(\cdot)$ disappear. This is because when only symbol extensions in time are applied, each channel element becomes a scaled identity matrix, see (4). Also, notice that the rank descriptors are now complex magnitudes. Therefore, the elimination matrix is a 3×4 full-row rank matrix having a one-dimensional null space. Consequently, the SSM becomes rank deficient, since there are some rank descriptors different from zero and desired signals cannot be linearly separated from interference.

APPENDIX C
PROOF OF LEMMA2

Due to similarity with the proof for $p = 2$, we elaborate a sketch of the proof for $p = 3$ and provide intuition of the proof for cases $p = 4, 5, 6$ by means of examples of its elimination matrices.

The SSM for $p = 3$ is constructed by using (37)-(38). As before, without loss of generality we consider receiver 1 only. In this case, after applying some full-rank linear transformations to the SSM, the following system of four equations is obtained:

$$\left[\mathbf{C}\left(h_{1,1}^{1,1}\right) - \mathbf{C}\left(\alpha_{1,1}^{\text{des}}\right)\right] \mathbf{A}_3 \boldsymbol{\lambda}_1 + \mathbf{C}\left(h_{1,1}^{1,2}\right) \mathbf{A}_2 \boldsymbol{\lambda}_2 + \left[\mathbf{C}\left(h_{1,1}^{1,3}\right) - \mathbf{C}\left(\alpha_{1,2}^{\text{des}}\right)\right] \mathbf{A}_1 \boldsymbol{\lambda}_3 + \mathbf{C}\left(h_{1,3}^{1,1}\right) \mathbf{A}_2 \boldsymbol{\lambda}_4 = \mathbf{0},$$

$$\left[\mathbf{C}\left(h_{1,1}^{2,1}\right) - \mathbf{C}\left(\alpha_{2,1}^{\text{des}}\right)\right] \mathbf{A}_3 \boldsymbol{\lambda}_1 + \mathbf{C}\left(h_{1,1}^{2,2}\right) \mathbf{A}_2 \boldsymbol{\lambda}_2 - \mathbf{C}\left(\alpha_1^{\text{int}}\right) \mathbf{A}_2 \boldsymbol{\lambda}_4 + \mathbf{A}_1 \boldsymbol{\lambda}_5 = \mathbf{0},$$

$$\mathbf{C}\left(h_{1,1}^{3,2}\right) \mathbf{A}_2 \boldsymbol{\lambda}_2 + \left[\mathbf{C}\left(h_{1,1}^{3,3}\right) - \mathbf{C}\left(\alpha_{3,2}^{\text{des}}\right)\right] \mathbf{A}_1 \boldsymbol{\lambda}_3 + \mathbf{A}_3 \boldsymbol{\lambda}_6 - \mathbf{C}\left(\alpha_2^{\text{int}}\right) \mathbf{A}_2 \boldsymbol{\lambda}_7 = \mathbf{0},$$

$$\left[\mathbf{C}\left(h_{1,1}^{41}\right) - \mathbf{C}\left(\alpha_{4,1}^{\text{des}}\right)\right] \mathbf{A}_3 \boldsymbol{\lambda}_1 + \mathbf{C}\left(h_{1,1}^{42}\right) \mathbf{A}_2 \boldsymbol{\lambda}_2 + \left[\mathbf{C}\left(h_{1,1}^{4,3}\right) - \mathbf{C}\left(\alpha_{4,2}^{\text{des}}\right)\right] \mathbf{A}_1 \boldsymbol{\lambda}_3 + \mathbf{C}\left(h_{1,2}^{4,3}\right) \mathbf{A}_2 \boldsymbol{\lambda}_7 = \mathbf{0},$$

where the SPBs are chosen as in (36), i.e:

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{B}_{1:2} & \mathbf{B}_{3:6} & \mathbf{B}_{7:8} \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} \mathbf{B}_{3:6} & \mathbf{B}_{9:12} \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{B}_{1:2} & \mathbf{B}_{9:12} & \mathbf{B}_{13:14} \end{bmatrix},$$

and

$$\alpha_{q,1}^{\text{des}} = \frac{h_{1,1}^{q,2} h_{3,1}^{3,1}}{h_{3,1}^{3,2}}, \quad \alpha_{q,2}^{\text{des}} = \frac{h_{1,1}^{q,2} h_{2,1}^{2,3}}{h_{2,1}^{2,2}},$$

$$\alpha_1^{\text{int}} = \frac{h_{1,3}^{2,2} h_{2,3}^{3,1}}{h_{2,3}^{3,2}}, \quad \alpha_2^{\text{int}} = \frac{h_{1,2}^{3,2} h_{3,1}^{2,3}}{h_{3,1}^{2,2}}.$$

A full-rank SSM is obtained as long as all rank descriptors $\boldsymbol{\lambda}_i, i = 1, \ldots, 7$ are equal to the zero vector. For instance, consider the elimination matrix in (53), obtained for the group Z (see (35)) after applying similar steps as in Appendix B, and equating imaginary parts to zero. Notice that this elimination matrix is full rank almost surely, since each row contains at least one element of the direct channel. Therefore, all rank descriptors involved in (53) can be set to zero.

Similar ideas apply to cases $p = 4, 5, 6$. For the sake of brevity, we show only the elimination matrix analogous to (53) for each of those cases at the next page, see (54)-(56). Following similar arguments discussed above, it can be ensured that all the elimination matrices are full rank, they have no right null space, and thus all involved rank descriptors can be set to zero. Note that to simplify notation we have used the function $\psi(a, b)$, defined as the sum of the sinusoidal functions corresponding to the position (a, b) of each elimination matrix.

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$$\begin{bmatrix} \left|h_{1,1}^{1,2}\right| \sin\left(\phi_{1,1}^{1,2}\right) & \left|h_{1,1}^{1,3}\right| \sin\left(\phi_{1,1}^{1,3}\right) - \left|\alpha_{1,2}^{\text{des}}\right| \sin\left(\alpha_{1,2}^{\text{des}}\right) & \left|h_{1,3}^{1,1}\right| \sin\left(\phi_{1,3}^{1,1}\right) & 0 \\ \left|h_{1,1}^{2,2}\right| \sin\left(\phi_{1,1}^{2,2}\right) & 0 & -\left|\alpha_{1}^{\text{int}}\right| \sin\left(\alpha_{1}^{\text{int}}\right) & 0 \\ \left|h_{1,1}^{3,2}\right| \sin\left(\phi_{1,1}^{3,2}\right) & \left|h_{1,1}^{3,3}\right| \sin\left(\phi_{1,1}^{3,3}\right) - \left|\alpha_{3,2}^{\text{des}}\right| \sin\left(\alpha_{3,2}^{\text{des}}\right) & -\left|\alpha_{1}^{\text{int}}\right| \sin\left(\alpha_{1}^{\text{int}}\right) & -\left|\alpha_{2}^{\text{int}}\right| \sin\left(\alpha_{2}^{\text{int}}\right) \\ \left|h_{1,1}^{42}\right| \sin\left(\phi_{1,1}^{42}\right) & \left|h_{1,1}^{4,3}\right| \sin\left(\phi_{1,1}^{4,3}\right) - \left|\alpha_{4,2}^{\text{des}}\right| \sin\left(\alpha_{4,2}^{\text{des}}\right) & \left|h_{1,2}^{4,3}\right| \sin\left(\phi_{1,2}^{4,3}\right) & 0 \end{bmatrix} \begin{bmatrix} \lambda_2(q) \\ \lambda_3(q) \\ \lambda_4(q) \\ \lambda_7(q) \end{bmatrix} = \mathbf{0} \tag{53}$$

$p = 4:$

$$\begin{bmatrix} 0 & \psi(1,2) & \psi(1,3) & 0 \\ \psi(2,1) & 0 & \psi(2,3) & 0 \\ 0 & \psi(3,2) & 0 & \psi(3,4) \\ \psi(4,1) & \psi(4,2) & \psi(4,3) & \psi(4,4) \\ \psi(5,1) & 0 & \psi(5,3) & \psi(5,4) \end{bmatrix} \begin{bmatrix} \lambda_1(q) \\ \lambda_3(q) \\ \lambda_4(q) \\ \lambda_9(q) \end{bmatrix} = \mathbf{0} \tag{54}$$

$p = 5:$

$$\begin{bmatrix} \psi(1,1) & 0 & 0 & 0 & \psi(1,5) & 0 \\ \psi(2,1) & \psi(2,2) & \psi(2,3) & \psi(2,4) & \psi(2,5) & 0 \\ 0 & \psi(3,2) & 0 & \psi(3,4) & \psi(3,5) & 0 \\ \psi(4,1) & 0 & \psi(4,3) & 0 & 0 & \psi(4,6) \\ \psi(5,1) & \psi(5,2) & \psi(5,3) & \psi(5,4) & 0 & \psi(5,6) \\ 0 & \psi(6,2) & 0 & 0 & 0 & \psi(6,6) \end{bmatrix} \begin{bmatrix} \lambda_1(q) \\ \lambda_2(q) \\ \lambda_4(q) \\ \lambda_5(q) \\ \lambda_9(q) \\ \lambda_{11}(q) \end{bmatrix} = \mathbf{0} \tag{55}$$

$p = 6:$

$$\begin{bmatrix} 0 & 0 & 0 & \psi(1,4) & \psi(1,5) & 0 & 0 \\ 0 & 0 & \psi(2,3) & 0 & \psi(2,5) & \psi(2,6) & 0 \\ \psi(3,1) & \psi(3,2) & \psi(3,3) & \psi(3,4) & \psi(3,5) & \psi(3,6) & 0 \\ 0 & \psi(4,2) & 0 & \psi(4,4) & 0 & \psi(4,6) & 0 \\ \psi(5,1) & 0 & \psi(5,3) & 0 & 0 & 0 & \psi(5,7) \\ \psi(6,1) & \psi(6,2) & \psi(6,3) & \psi(6,4) & 0 & 0 & \psi(6,7) \\ 0 & \psi(7,2) & 0 & \psi(7,4) & 0 & 0 & \psi(7,7) \end{bmatrix} \begin{bmatrix} \lambda_2(q) \\ \lambda_3(q) \\ \lambda_5(q) \\ \lambda_6(q) \\ \lambda_7(q) \\ \lambda_8(q) \\ \lambda_{13}(q) \end{bmatrix} = \mathbf{0} \tag{56}$$

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