ASYMPTOTIC ANALYSIS OF SYNCHRONOUS SIGNAL PROCESSING

Signal Processing & Communications

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Highlights

- Asymptotic **Karhunen–Loève integral expansion of cyclostationary (CS) processes** and connection to their Cramér–Loève representation.
- Extension of classical stationary Wiener filtering results.
- **Unified theoretical analysis** of smoothing, filtering and prediction of CS processes.
- Quantification of **minimum mean squared error and achievable synchronous gain** in terms of coherence statistics.

Let $\{x(n) \in \mathbb{C}\}_{n\in\mathbb{Z}}$ be a zero-mean, proper and second-order random process with autocorrelation function

$$\mathrm{R}_x(n,m) \triangleq \mathrm{E}[x(n+m)x^*(n)].$$

Types of processes:

Wide-sense stationary (WSS): $\mathrm{R}_x(n,m) \equiv \mathrm{R}_x(m)$.

Cyclostationary (CS): $\mathrm{R}_x(n,m) \equiv \mathrm{R}_x(n+lP,m) \quad \forall l \in \mathbb{Z}$.

Cramér–Loève (CL) expansion

$$x(n) = \int_0^1 \mathrm{e}^{\mathrm{j}2\pi f n}\mathrm{d}\nu_x(f)$$
$$\mathrm{d}\nu_x(f) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} x(n)\mathrm{e}^{-\mathrm{j}2\pi f n} \triangleq \mathrm{CL}\{x(n)\}(f). \quad (1)$$

Spectral correlation:

$$\mathrm{S}_x(\alpha,f)\mathrm{d}\alpha\mathrm{d}f \triangleq \mathrm{E}\left[\mathrm{d}\nu_x(f)\mathrm{d}\nu_x^*(f-\alpha)\right].$$

- **WSS processes** have orthogonal increments:
$$\mathrm{S}_X^{(\mathrm{WSS})}(\alpha,f) = \mathrm{S}_X(f)\,\delta(\alpha),$$
where $\mathrm{S}_X(f)\mathrm{d}f \triangleq \mathrm{E}[|\mathrm{d}\nu_x(f)|^2]$ is the power spectral density (PSD).
- **CS processes** display spectral lines:
$$\mathrm{S}_X^{(\mathrm{CS})}(\alpha,f) \triangleq \sum_{k=0}^{P-1}\mathrm{S}_X^{(k/P)}(f)\,\delta\left(\alpha-\tfrac{k}{P}\right),$$
where
$$\mathrm{S}_X^{(k/P)}(f) \triangleq \frac{1}{P}\sum_{n=0}^{P-1}\sum_{m\in\mathbb{Z}}\mathrm{R}_x(n,m)\mathrm{e}^{-\mathrm{j}2\pi\left(\frac{kn}{P}+fm\right)}$$
is the *cyclic PSD*.

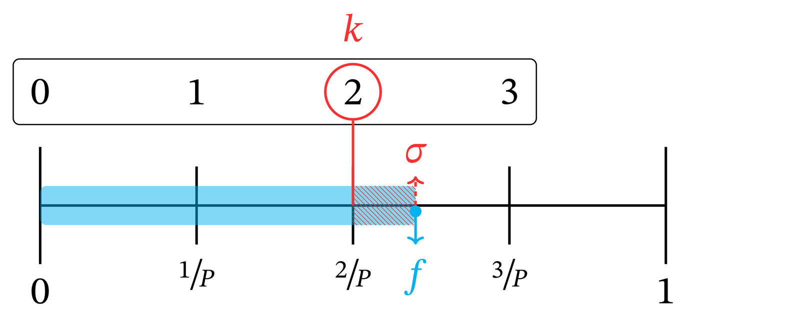

Figure 1. Spectral domain segmentation: $f \triangleq \sigma + k/P$.

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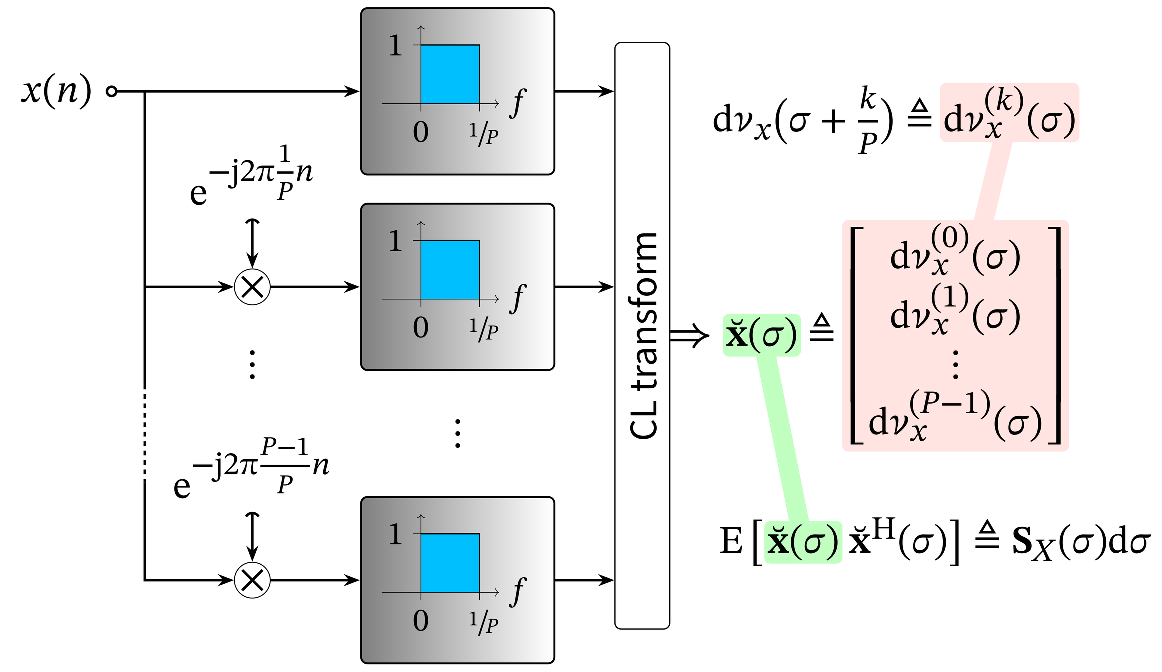
Figure 2. Gladyshev decomposition (harmonic series representation).



Karhunen–Loève (KL) expansion

$$x(n) = \int_0^1 \phi(n,\lambda)\mathrm{d}\xi_x(\lambda)$$
$$\mathrm{d}\xi_x(\lambda) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} x(n)\phi^*(n,\lambda) \triangleq \mathrm{KL}\{x(n)\}(\lambda). \quad (2)$$

Basis $\{\phi(n,\lambda)\}$ solves the following eigenequation:

$$\sum_{l\in\mathbb{Z}}\mathrm{R}_x(l,k-l)\phi(l,\lambda) = \mathcal{S}_X(\lambda)\phi(k,\lambda),$$

where $\mathcal{S}_X(\lambda)\mathrm{d}\lambda \triangleq \mathrm{E}[|\mathrm{d}\xi_x(\lambda)|^2]$ is the *KL spectrum* of $\{x(n)\}$.

- KL basis of WSS processes is $\{\phi_{\mathrm{WSS}}(n,\lambda) \triangleq \mathrm{e}^{\mathrm{j}2\pi\lambda n}\}$.

Theorem 1 – KL basis of CS processes

The Karhunen–Loève eigenbasis of a P-periodic cyclostationary process is

$$\phi_{\mathrm{CS}}^{(p)}(n,\sigma) = \sum_{q=0}^{P-1}\left[\mathbf{B}_X(\sigma)\right]_{q,p}\mathrm{e}^{\mathrm{j}2\pi\left(\sigma+\frac{q}{P}\right)n}. \quad (3)$$

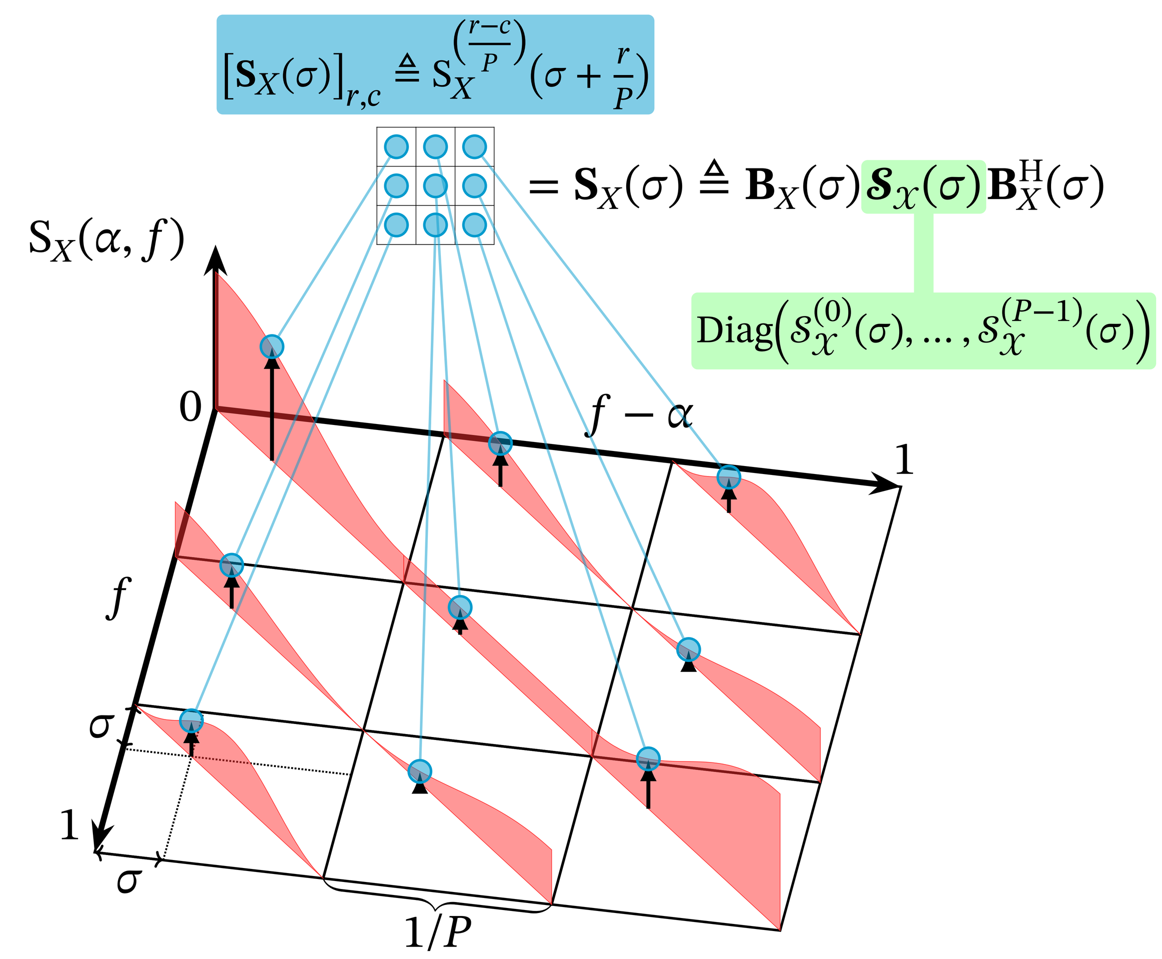


Figure 3. Construction of the cyclic spectrum matrix (CSM).

Corollary 1: KL–CL connection

The KL and CL spectral representations of a CS process are related through the following inequality:

$$\check{\mathbf{x}}(\sigma) = \mathbf{B}_X^{\mathrm{H}}(\sigma)\check{\mathbf{x}}(\sigma), \quad (4)$$

where $\check{\mathbf{x}}(\sigma) \triangleq \left[\mathrm{d}\xi_x^{(0)}(\sigma),\dots,\mathrm{d}\xi_x^{(P-1)}(\sigma)\right]^{\mathrm{T}}$.

Optimalities of the KL representation:

Signal model:

$x(n) = d(n-\varepsilon)+z(n) \Rightarrow$
- $\{d(n)\}$: digital PAM signal.
- $\{z(n)\}$: additive white Gaussian noise.
- $\varepsilon \sim \mathcal{U}[0,\Delta)$: synchronization reliability.

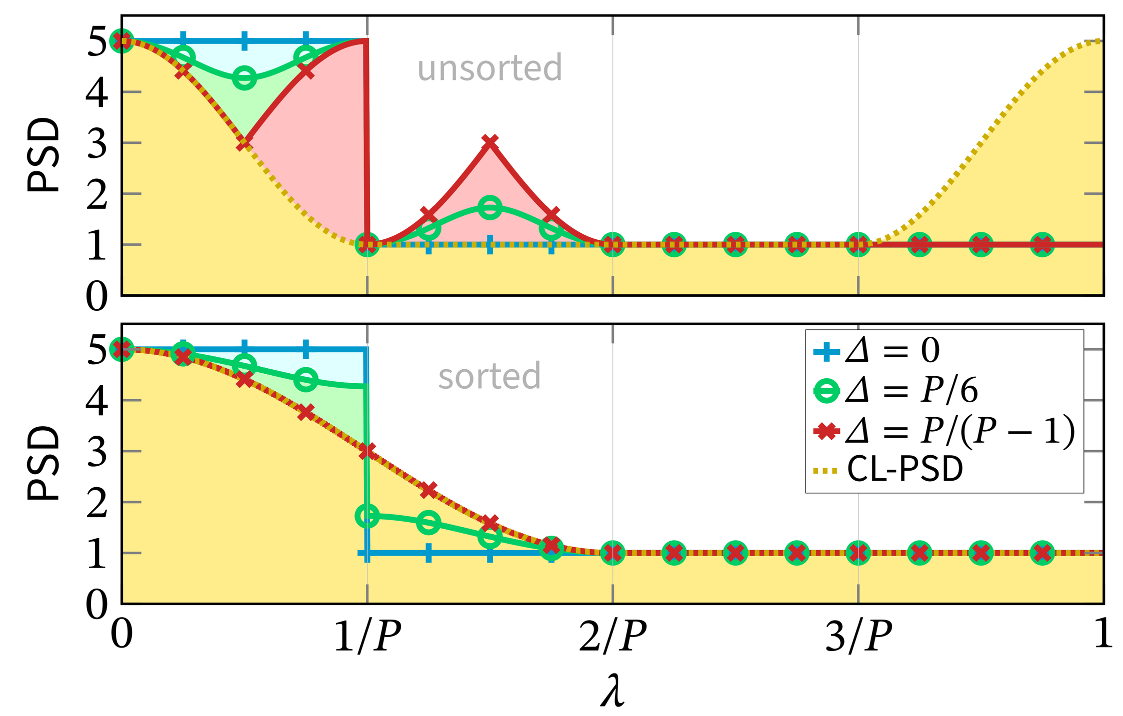


Figure 4. Optimal energy compaction of the KL representation.

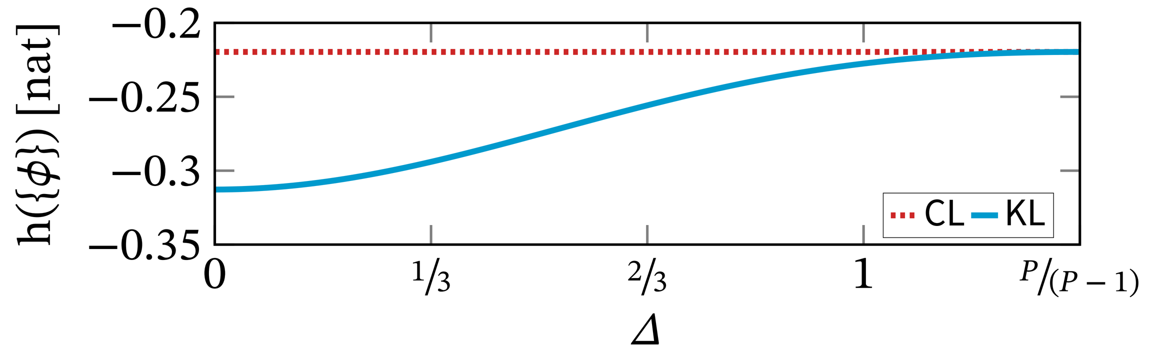


Figure 5. Minimum representation entropy of the KL representation.

Cyclic Wiener filtering

Smoothing

Theorem 2 – Cyclic Wiener filter in the KL domain

The (noncausal) cyclic Wiener filter is equivalent to the KL Wiener filter.

- The FRESH implementation can be expressed as a multiplicative filter in the KL domain: $\mathcal{W}_p(\sigma) = \mathrm{E}[\mathrm{d}\xi_d^{(p)}(\sigma)\mathrm{d}\xi_x^{(p)*}(\sigma)]\ /\ \mathrm{E}[|\mathrm{d}\xi_x^{(p)}(\sigma)|^2]$.
- Minimum MSE:
$$\mathrm{MMSE}_{\mathrm{nc}} = \int_0^{1/P}\mathrm{Tr}\left[\mathbf{S}_D(\sigma)(\mathbf{I}_P - \mathbf{C}_{DX}(\sigma)\mathbf{C}_{DX}^{\mathrm{H}}(\sigma))\right]\mathrm{d}\sigma, \quad (5)$$
where $\mathbf{C}_{DX}(\sigma) \triangleq \mathbf{S}_D^{-1/2}(\sigma)\,\mathrm{E}[\check{\mathbf{d}}(\sigma)\check{\mathbf{x}}^{\mathrm{H}}(\sigma)]\mathbf{S}_X^{-1/2}(\sigma)$ is the *coherence matrix*.
- High SNR regime: $\mathrm{MMSE}_{\mathrm{nc}} \xrightarrow{\mathrm{SNR}\to\infty} B/\mathrm{SNR}$, where $\mathcal{S}_D(\lambda)$ is defined for $\lambda\in[0,B)$.

Filtering

- Derived using the *Guo–Shamai–Verdú theorem*:
$$\mathrm{MMSE_c(SNR)} = \mathrm{E}[\mathrm{MMSE_{nc}}(\Gamma \sim \mathcal{U}[0,\mathrm{SNR}])]$$
$$= -\frac{1}{\mathrm{SNR}}\int_0^{1/P}\ln|\mathbf{I}_P - \mathbf{C}_{DX}(\sigma)\mathbf{C}_{DX}^{\mathrm{H}}(\sigma)|\mathrm{d}\sigma. \quad (6)$$
- High SNR regime: $\mathrm{MMSE_c(SNR)} \xrightarrow{\mathrm{SNR}\to\infty} B\frac{\ln\mathrm{SNR}}{\mathrm{SNR}} + O(1/\mathrm{SNR})$.

Prediction

- MMSE of one-step prediction of $\{x(n)\}$:
$$\mathrm{MMSE_p} = \exp\int_0^{1/P}\ln|\mathbf{S}_X(\sigma)|\mathrm{d}\sigma. \quad (7)$$
- High SNR regime: $\mathrm{MMSE_p(SNR)} \xrightarrow{\mathrm{SNR}\to\infty} A/\mathrm{SNR}^{(1-B)}$.
- Synchronous gain ($\propto$ *spectral flatness*):
$$\zeta_{\mathrm{P}} = \frac{\mathrm{MMSE_p}}{\mathrm{MMSE_{WSS,p}}} = \exp\int_0^{1/P}\ln|\overline{\mathbf{C}}_X(\sigma)|\mathrm{d}\sigma, \quad (8)$$
where $\overline{\mathbf{C}}_X(\sigma) \triangleq (\mathbf{I}_P\circ\mathbf{S}_X^{-1/2}(\sigma))\mathbf{S}_X(\sigma)(\mathbf{I}_P\circ\mathbf{S}_X^{-1/2}(\sigma))$ is the *spectral coherence matrix*.

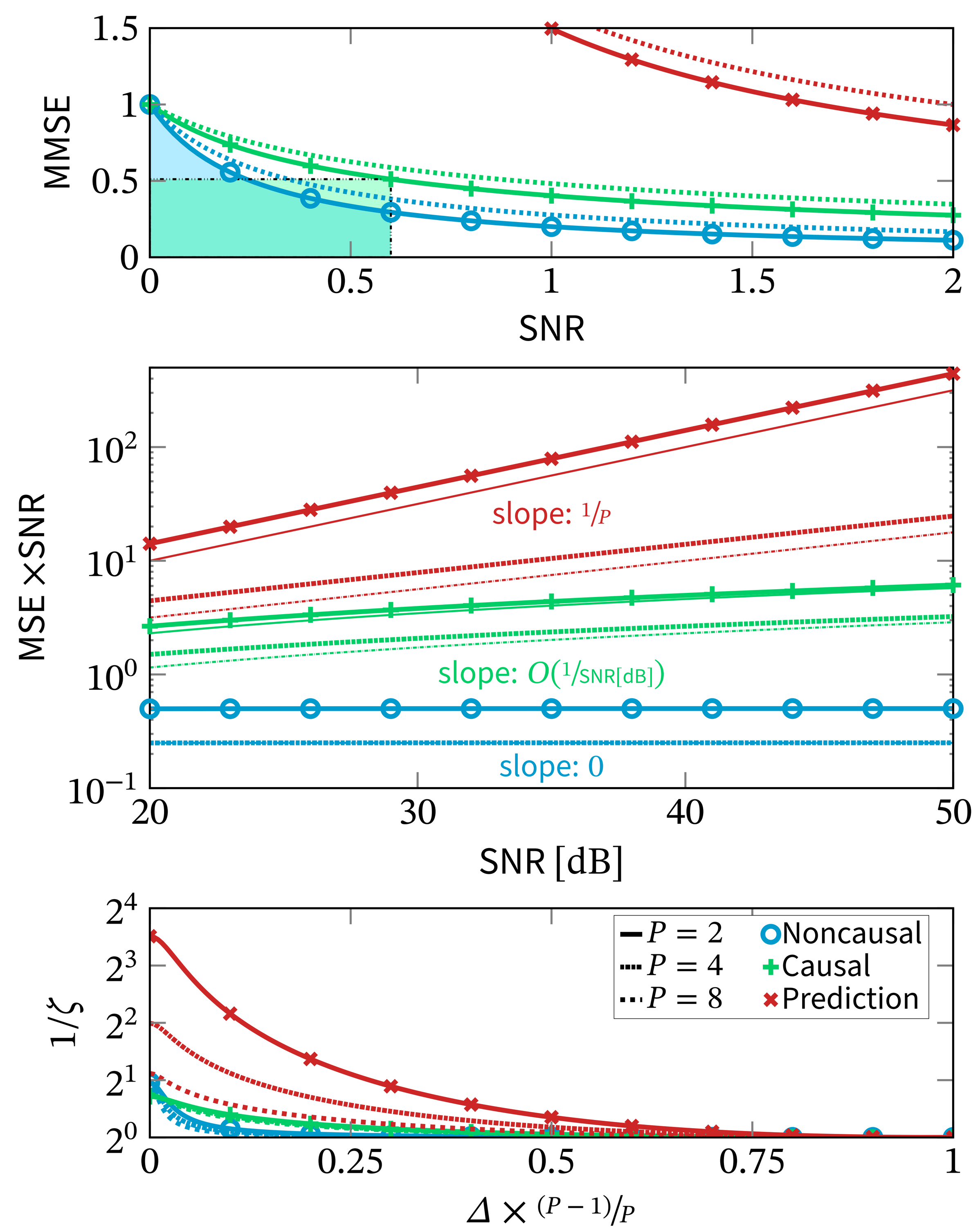


Figure 6. MMSE and synchronous gain illustrations.

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