Degrees of Freedom of the Rank-Deficient MIMO X channel

Adrian Agustin and Josep Vidal
Department of Signal Theory and Communications
Universitat Politècnica de Catalunya (UPC)
Barcelona, Spain
Email: {adrian.agustin, josep.vidal}@upc.edu

Abstract—The multiple-input multiple-output (MIMO) X channel with rank-deficient channels is considered. We characterize the total degrees of freedom (DoF) by defining an outer bound which is attained by the proposed precoding scheme. The total DoF are described in closed-form when all transmitters and receivers have \( M \) and \( N \) antennas, respectively. For an arbitrary number of antennas, the total DoF are obtained as solution of a linear programming problem. Results elucidate that, unlike point-to-point MIMO systems where DoF increase with the rank of the channel, the total DoF in a multipoint-to-multipoint MIMO system can increase up to a certain point if channels are rank-deficient. Hence, we corroborate the relevance of knowing the rank-deficiency in the MIMO X channel, a property already observed in the MIMO Interference channel.

I. INTRODUCTION

The degrees of freedom (DoF) is a system performance metric that characterizes how the system sum rate scales in the high power regime. For the most common type of multiuser multiple-input multiple output (MIMO) channels, like MIMO multiple access channel (MAC), MIMO broadcast (BC) and two-users MIMO interference channel (IC), the optimal DoF are obtained by means of spatial zero forcing linear filters [1]. However, for the three-users MIMO IC, one way to get the optimal DoF is by means of the interference alignment (IA) concept. Using IA concept, the transmit filters are designed in such a way that each receiver observes all interfering signals overlapped in a common subspace, while the desired signal is present in a different subspace, see for example [2].

The two-users MIMO X channel generalizes the two-by-two users communication. The MIMO MAC, BC and IC channels can be seen as particular cases, but the MIMO X has a superior performance in terms of DoF. This channel is illustrated in Fig. 1 when all terminals have \( M = 3 \) antennas. The \( j \)-th transmitter has independent messages to each receiver \((W_{1j}, W_{2j})\), and the \( i \)-th receiver has to decode messages \( W_{1i}, W_{2i} \). Precoders are designed for aligning the interfering signals (dotted lines) in a common subspace at unintended receivers, while intended signals (solid lines) lie in a different subspace.

The DoF of the MIMO X channel has been investigated in [3], [4]. In [4] it is shown that the total DoF is \( 4M/3 \) when all terminals are equipped with \( M \geq 1 \) antennas and channel coefficients remain fixed during the communication. Nevertheless, this latter result becomes true for \( M = 1 \) in case asymmetric complex signaling concept is exploited, [5]. On the other hand, the total DoF with arbitrary number of antennas are characterized in [6]. The precoding scheme is based on the Generalized Singular Value Decomposition (GSVD) which naturally provides the generating basis for designing the precoders with the objective of aligning the unintended transmissions or avoiding interference through null-steering (NS) transmission. This approach is also considered in [7].

The previous literature on DoF usually assumes full-rank channel matrices associated to a rich scattering propagation scenario. Nevertheless, this is not always the case for wireless networks where poor scattering, line-of-sight (LOS) situations or keyhole channels might produce rank-deficient channels, [8], [9]. For example, keyhole channels might come up in specific roof-top scenarios. Rank-deficient MIMO IC channels are investigated in [10],[11], showing that the attained DoF can be larger than in the full-rank channel case.

In this work we look into the effect of the rank-deficiency over the total DoF of the multipoint-to-multipoint MIMO X channel. Although rank-deficiency limits the spatial dimensions to be employed for intended transmissions, it also provides spatial dimensions that transmitters can use without generating interference at unintended receivers. Hence, we cannot anticipate if rank-deficiency improves or degrades the total DoF of the system. The main contributions are:

- We derive outer bound regions of DoF for rank-deficient MIMO Z and X channels in Theorem 1 and Theorem 2.

![Fig. 1. MIMO X channel. Transmitting and receiving terminals equipped with M = N = 3 antennas. Solid lines denote the intended signals, while dotted lines stand for the generated interference. Particular cases: BC if W22 = W12 = 0, MAC if W22 = W21 = 0 and IC when W21 = W12 = 0.](image-url)
Additionally, the outer bound on the total DoF for the MIMO $X$ channel is given in Corollary 1.

- Using the achievable scheme proposed in [6] and properly optimizing the number of symbol streams and symbol extensions, we attain the outer bound of the rank-deficient channel (see Theorem 3).
- We explicitly prove the case where all transmitters and receivers have $M$ and $N$ antennas, respectively (section V.A and Corollary 2). In other cases, the tightness of the outer bound is confirmed by Monte-Carlo simulations.

**Notation:** The operator $\otimes$ denotes the Kronecker product, rank returns the rank of a matrix, span describes the space generated by the columns of a matrix, operator $[A]_i^i$ selects the columns of matrix $A$ denoted by vector $[1 : 1 : i]$, the operators $\mathbb{R}\{\}$ and $\mathbb{S}\{\}$ returns the real and imaginary parts of a complex number. Finally, $C, \mathbb{R}, \mathbb{Z}$ are the complex, real and integer numbers.

**II. SYSTEM MODEL**

In the two-users MIMO $X$ channel shown in Fig. 1 the transmitters $T_1, T_2$ are equipped with $M_1, M_2$ antennas while receivers $R_1, R_2$ have $N_1, N_2$ antennas. The channel coefficients remain constant over the whole communication. We assume that transmissions are carried out over $2T$ channel uses thanks to a $T$ symbol extension with asymmetric complex signaling, [5]. Hence the equivalent channel is defined by

$$\hat{H}_{ij} = I_T \otimes \tilde{H}_{ij}, \quad \tilde{H}_{ij} = \begin{bmatrix} \mathbb{R}\{H_{ij}\} \\ \mathbb{S}\{H_{ij}\} \end{bmatrix}$$

(1)

where $T$ denotes the symbol extension, $H_{ij} \in \mathbb{C}^{N_j \times M_i}$ is the channel matrix between the $i$-th receiver and the $j$-th transmitter, $\tilde{H}_{ij} \in \mathbb{R}^{2N_j \times 2M_i}$ stands for the channel matrix after applying the asymmetric complex signaling and finally $\hat{H}_{ij} \in \mathbb{R}^{2T \times 2T}$ is the equivalent channel matrix when both type of channel extensions are adopted. The received signal at the $i$-th receiver is modeled by

$$\tilde{y}_i = \hat{H}_{i1}(\tilde{x}_{11} + \tilde{x}_{21}) + \hat{H}_{i2}(\tilde{x}_{12} + \tilde{x}_{22}) + \tilde{n}_i, \quad i = 1, 2$$

(2)

where $\tilde{n}_i \in \mathbb{R}^{2T \times 1}$ denotes the received noise, which is assumed additive white Gaussian noise (AWGN). Finally the input vectors at the transmitters have the following structure according to [6], [7]

$$\tilde{x}_{ij} = \left[ \tilde{V}_{ij} \tilde{Z}_{ij} \right] \begin{bmatrix} s_{ij}^A \\ s_{ij}^B \end{bmatrix}$$

(3)

where $\tilde{V}_{ij} \in \mathbb{R}^{2T \times M_i s_{ij}^A}$, $\tilde{Z}_{ij} \in \mathbb{R}^{2T \times M_i s_{ij}^B}$ are the interference alignment (IA) and null-steering (NS) precoders that transmit Gaussian symbol streams conveying message $W_{ij}$. These precoders are designed according to the following conditions,

$$\hat{H}_{ij} \tilde{Z}_{kj} = 0, \quad i, k, j = 1, 2 \quad i \neq k$$

(4)

$$\text{span}\left(\hat{H}_{ij} \tilde{V}_{ki}\right) \subseteq \text{span}\left(\hat{H}_{i2} \tilde{V}_{k2}\right)$$

(5)

where it is assumed wlog $d_{k2}^A \geq d_{k1}^A$. Notice that there are up to four possible cases with $(d_{12}^A, d_{11}^A)$ and $(d_{22}^A, d_{21}^A)$. The interference at the $i$-th receiver is given by

$$\tilde{I}_i = [\hat{H}_{i1} \tilde{V}_{k1} \hat{H}_{i2} \tilde{V}_{k2}] \begin{bmatrix} s_{k1}^A \\ s_{k2}^A \\ i \neq k, \ i, k = 1, 2 \end{bmatrix}$$

(6)

and the signal space matrix at the $i$-th receiver containing desired and interference signals becomes

$$G_i = [\hat{H}_{i1} \tilde{V}_{i1} \hat{H}_{i2} \tilde{V}_{i2} \hat{H}_{i1} \tilde{Z}_{i1} \hat{H}_{i2} \tilde{Z}_{i2}]$$

(7)

Assuming that achievable rate and capacity for each message are defined in the standard Shannon sense, the DoF attained with linear receivers are given by

$$\tilde{d}_{ij} = \frac{1}{2T} (d_{ij}^A + d_{ij}^B)$$

(8)

Once the transmit filters are designed, the receive filters get rid of interference by projecting the received signal onto the orthogonal space of the received interference.

This signal model is also valid for analyzing the reciprocal MIMO $X$ channel, which is defined by switching the direction of communication. For MIMO IC, [12], and MIMO $X$ channels, [6], the DoF are the same in original and reciprocal networks. We will consider this property for proposing an efficient mechanism to design the transmit filters based on the conditions defined in (3)-(5). According to them, for a configuration $M_1 = M_2 = 2$ and $N_1 = N_2 = 5$, we cannot exploit neither NS nor IA transmission in the original network. We might use the IA concept in case we remove 2 receiving antennas. In contrast, this work will analyze the reciprocal network (i.e. 5 transmitting antennas and 2 receiving antennas).

**III. OUTER BOUNDS ON DEGREES OF FREEDOM**

The MIMO $Z$ channel, analyzed in [4], presents the same input-output equations as the MIMO $X$ channel, but MIMO $Z(ij)$ channel is obtained when message $W_{ij}$ is removed and the channel associated to the link for transmitting that message has zero gain $(W_{ij} = 0, \ H_{ij} = 0)$. The outer bound DoF region for this channels is defined in [4] for full-rank channels. In the following we define the outer bound DoF for the rank-deficient case.

**Theorem 1:** An outer bound DoF region for the $Z(ij)$ channel is defined as follows

$$D_{out}^{Z(ij)} = \left\{ (\tilde{d}_{11}, \tilde{d}_{12}, \tilde{d}_{21}, \tilde{d}_{22}) \in \mathbb{R}^4, \tilde{d}_{ij} = 0 \right\}$$

(9)

$$\tilde{d}_{11} + \tilde{d}_{12} + \tilde{d}_{21} + \tilde{d}_{22} \leq N_k + M_q - r_{kq}, \quad \tilde{d}_{n1} + \tilde{d}_{n2} \leq N_n,$$

$$\tilde{d}_{1m} + \tilde{d}_{2m} \leq M_m, \quad \tilde{d}_{nm} \leq r_{nm}, \quad n, m = 1, 2, \ k \neq i, q \neq j$$

where $\tilde{d}_{nm}$ denotes the DoF that carries message $W_{nm}$, and $r_{kq} = \text{rank} (H_{kq})$. Let us remark that the channel matrix is not required to be full-rank.

**Proof:** The latter three constraints in (9) are associated to the MAC and BC channels present in the MIMO $Z(ij)$ channel and the available transmission modes at the different channel matrices. The first constraint in (9) is obtained by
the application of Theorem 1 in [4] to the MIMO $Z(ij)$ rank-deficient channel, where we add $M_q - r_{kq}$ antennas at receiver $R_q$, with $i \neq k$, $q \neq j$. The channel coefficients of the new additional antennas are drawn i.i.d. from a continuous distribution, and thus, the new channel matrix describing the link between transmitter $T_q$ and receiver $R_q$ is full rank. ■

Since MIMO $X$ channel subsumes multiple $Z$ channels, Theorem 2: An outer bound DoF region for the MIMO $X$ channel is defined as

$$D_{\text{out}}^X = D_{\text{out}}^{(11)} \cap D_{\text{out}}^{(12)} \cap D_{\text{out}}^{(21)} \cap D_{\text{out}}^{(22)}$$

$$= \left\{ (\hat{d}_{11}, \hat{d}_{12}, \hat{d}_{21}, \hat{d}_{22}) \in \mathbb{R}^4, \right.$$  

$$\left. \begin{array}{l}
\hat{d}_{11} + \hat{d}_{12} + \hat{d}_{21} \leq N_1 + M_1 - r_{11}, \\
\hat{d}_{11} + \hat{d}_{21} + \hat{d}_{22} \leq N_2 + M_1 - r_{12}, \\
\hat{d}_{12} + \hat{d}_{21} + \hat{d}_{22} \leq N_1 + M_2 - r_{12}, \\
\hat{d}_{n1} + \hat{d}_{n2} \leq N_n, \\
\hat{d}_{1m} + \hat{d}_{2m} \leq M_m \quad n, m = 1, 2 \right\}$$

Proof: Applying Theorem 1 to the different MIMO $Z$ channels present in a MIMO $X$ channel. ■

Corollary 1: The outer bound on the total DoF for a MIMO $X$ channel is defined by,

$$\eta_{\text{out}}^X \triangleq \max_{d_{en} \in D_{\text{out}}^X} \left( \hat{d}_{11} + \hat{d}_{12} + \hat{d}_{21} + \hat{d}_{22} \right)$$

Proof: Maximization of the sum DoF over the outer bound region (linear constraints) defined by Theorem 2. ■

Corollary 2: The outer bound on the total DoF for a MIMO $X$ channel when both transmitters are equipped with $M$ antennas, receivers have $N$ antennas and $r_d = \text{rank}(H_{ij})$ for $i = 1, 2$ and $j \neq i$ is given by,

$$\eta_{\text{out}}^X \triangleq \left\{ \begin{array}{ll}
2\gamma & \text{if } \gamma \leq K_{\min}, \\
2K_{\min} & \text{if } K_{\min} < \gamma \leq 2K_{\max} - K_{\min}, \\
\frac{4}{3} \left( M + N - \frac{\gamma}{2} \right) & \text{if } 2K_{\max} - K_{\min} < \gamma.
\end{array} \right.$$ (12)

where $\gamma = r_d + r_c$, $K_{\min} = \min(M, N)$, $K_{\max} = \max(M, N)$.

Proof: The linear programming problem in (11) is solved using standard tools, [13], obtaining a closed-form solution. ■

IV. ACHIEVABLE MIMO SCHEME

We adopt the GSVD-based precoding scheme proposed in [6] for the full-rank MIMO $X$ channel, where the transmit precoders for messages originated at transmitter $T_j$ and intended to receiver $R_i$ are selected as,

$$\hat{V}_{ij} = \left[ I_T \otimes \hat{\Omega}_{ij} \right] \left[ \begin{array}{c}
T_1 \\
\vdots \\
T_T
\end{array} \right]^{1:d_{ij}^N}, \hat{Z}_{ij} = [I_T \otimes \hat{\Psi}_{ij}]^{1:d_{ij}^N}$$ (13)

where $\hat{V}_{ij} \in \mathbb{R}^{2MT_j \times d_{ij}^N}$, $\hat{Z}_{ij} \in \mathbb{R}^{2MT_j \times d_{ij}^N}$ are the precoders that exploit the IA and NS concepts, respectively. Moreover, $\hat{\Omega}_{ij} \in \mathbb{R}^{2MT_j \times 2d_{ij}^N}$, $\hat{\Psi}_{ij} \in \mathbb{R}^{2TM_j \times 2d_{ij}^N}$ with $i \neq k$ are the overlapping and null-steering basis obtained using the GSVD decomposition of channel matrices $(H_{k1}, H_{k2})$ (see [6], [7]).

The dimensions of the previous basis depends on,

$$s_k = \text{rank}(H_{k1}) + \text{rank}(H_{k2}) - \text{rank}([H_{k1} \ H_{k2}])$$ (14)

$$\phi_{ij} = M_j - \text{rank}(H_{ij})$$ (15)

Finally, matrices $T_{ni} \in \mathbb{R}^{2N_i \times 2d_{nk}}$ are independently drawn with random elements [6, section VI].

Lemma 1 (Symbol Stream Optimization): Given a $2T$ channel matrix obtained using asymmetric complex signaling and $T$-symbol extension, the number of transmitted symbol streams per message $(d_{ij}^A, d_{ij}^N)$ are obtained as the solution of the following integer linear programming which maximizes the sum achievable DoF of the MIMO $X$ channel

$$\zeta(T) \triangleq \max_{\{d_{ij}^A\}, \{d_{ij}^N\}} \left( \frac{1}{2T} \sum_{i,j} d_{ij}^A + d_{ij}^N \right)$$ (16)

subject to

$$\Delta(T) \triangleq \left\{ (d_{ij}^A, d_{ij}^N) : \right.$$  

$$\left. \begin{array}{l}
d_{ij}^A + d_{ij}^N + d_{11}^A + d_{12}^A + d_{21}^A + d_{22}^A \leq 2TN_i, \\
d_{ij}^N + d_{ij}^A + d_{12}^A + d_{21}^A + d_{22}^A \leq 2TN_i, \\
d_{ij}^A + d_{ij}^N + d_{12}^A + d_{21}^A + d_{22}^A \leq 2TM_j, \\
d_{ij}^A \leq s_k, \\
d_{ij}^A, d_{ij}^N \leq 2T\phi_{ij}, \\
d_{ij}^A + d_{ij}^N \leq 2T \text{rank}(H_{ij}) \quad i, k, j = 1, 2, i \neq k \right\}$$ (17)

Proof: Notice that $\Delta(T)$ defines the set of constraints that must be satisfied by the number of transmitted symbol streams $(d_{ij}^A, d_{ij}^N)$ due to the number of receiving/transmitting antennas, the dimension of the overlapped space ($s_k$ in (14)) and the null-steering space ($\phi_{ij}$ in (15)). ■

Theorem 3 (Achievable sum DoF): The achievable sum DoF for the 2-user MIMO $X$ channel with $M_1$, $M_2$ transmitting antennas and $N_1$, $N_2$ receiving antennas when precoding is based on NS and IA over a constant channel is given by

$$\eta_{\text{out}}^X = \max_T \left( \zeta(T) \right)$$ (18)

where $\zeta(T)$ is maximum sum DoF when solving the integer linear programming problem defined in Lemma 1 for a given $T$-symbol extension, while $\zeta(T)$ is the obtained solution when we optimize the transmitters and receivers for reciprocal network, i.e., with antenna configuration $M = N = M$.■

Proof: We optimize (18). If the best solution is met in the reciprocal network, then the obtained transmit (resp. receive) filters have to be used as receive (resp. transmit) filters in the original network. ■

V. TOTAL DOF

The total DoF of the MIMO $X$ channel is attained by the scheme presented in the previous section when the number of symbol streams is optimized using Theorem 3. In such a case, the sum DoF coincides with the outer bound presented
in Corollary 1. Note that the full-rank case is a particular case, [6]. The analysis with rank-deficiency implies increasing the number of involved variables (antenna elements, $T$ symbol extension, number of streams using NS or IA concepts, and rank of the different channel matrices) and it is difficult to prove explicitly the optimality of the achievable scheme. In the following we solve two particular cases:

A. Symmetric antenna and rank channel configuration

The number of symbol streams per message when transmitters and receivers have $M$ and $N$ antennas, respectively, and the rank of channels is given by $r_d=\text{rank}(H_{11})=\text{rank}(H_{22})$, $r_c=\text{rank}(H_{12})=\text{rank}(H_{21})$ are equal to,

$$d^A = d^A_1 = d^A_{11} = d^A_{21} = d^A_{22},$$
$$d^{NS} = d^{NS}_{11} = d^{NS}_{22},$$
$$d^c = d^c_{12} = d^c_{21}$$

In case $M>N$, the largest sum DoF are obtained in the original MIMO X network, while reciprocal network becomes relevant when $M<N$. Notice that in this latter case, there are not overlapped ($s_1=s_2=0$ in (14)) or null-steering ($\theta_{ij}=0$ in (15)) spaces in the original network. The total DoF is obtained as the solution of the following optimization problem,

$$\eta^X_{\alpha} = \text{maximize} \quad \max_{d^A, d^{NS}, d^c} \left\{ \frac{1}{2T} \left( 4d^A + 2d^{NS} + 2d^c \right) \right\}$$

subject to,

$$d^{NS} + d^c + 3d^A \leq 2TN,$$
$$d^A + d^{NS} + 2d^c \leq 2TM,$$
$$d^A \leq 2\max \{ r_d + r_c - N, 0 \},$$
$$d^A \leq 2T(M-r_c),$$
$$d^c \leq 2T(M-r_d),$$
$$d^A + d^{NS} \leq 2Tr_d,$$
$$d^A + d^c \leq 2Tr_c,$$
$$(d^A, d^{NS}, d^c) \in \mathbb{Z}^3$$

The outer bound given in Corollary 2 is tightly attained by solving the optimization problem in (19) for $T=3$. In case $r_c+r_d\leq N$, the total DoF are equal to $2(r_c+r_d)$ and can be attained by using $d^A = 6r_d$, $d^{NS} = 6r_c$, $d^c = 0$. On the other hand, when $r_c+r_d>N$, Table I and Table II present the total attained DoF as a function of the values of $r_c$ and $r_d$, along with the optimal number of symbol streams per message.

Fig 2 compares the performance of the MIMO X channel with the cooperative broadcast channel, i.e. one transmitter with $2M$ transmitting antennas serving two users with $N$ antennas each, and the two-users interference channel analyzed in [10]. Results are presented as total DoF as a function of $r_d$ when $N \leq M \leq 3N/2$ and $r_c \leq \frac{2}{3}(M-N)$. We can observe that full cooperation among transmitters is useless from a DoF standpoint when $r_d \leq 2M-N-r_c$. Otherwise, when $r_d > 2M-N-r_c$, the DoF performance of the MIMO X channel worsens, but it gets a better performance than the interference channel because in the MIMO X channel each transmitter has messages intended to all receivers. Finally, let us remark that total DoF of the rank-deficient MIMO X channel always is superior to the full-rank case whenever $N-r_c < r_d < 2M-N-r_c$.

Finally, Fig. 3 depicts of sum-rate as a function of the signal to noise ratio (SNR) with $M_1 = M_2 = 5$ and $N_1 = N_2 = 4$ antennas, and different values for the rank of channels ($r_d$, $r_c$). The obtained DoF coincide with the predicted values and the outer bound of Corollary 2.

B. Asymmetric antenna and rank channel configuration

We have evaluated numerically a large number of antenna and rank channel configurations and the attained total DoF by Theorem 3 coincide with the outer bound described by Corollary 1. Table III presents some results when $M_1 = 4$, $M_2 = 3$, $N_1 = 3$ and $N_2 = 5$. In order to illustrate the role of the reciprocal network, cases A and B in Table III analyze the situation.
the same configuration. The former one optimizes the symbol streams in the original MIMO X network, while case B solves the reciprocal network. The outer bound is attained in case B. The remaining cases (C, D, E), attain the outer bound in the reciprocal network. The outer bound is attained in case B.

VI. CONCLUSION

We have characterized the total DoF for the rank-deficient MIMO X channel. The optimal total DoF are achieved by using a precoding scheme based on the interference alignment and null-steering concepts, along with the proper optimization of the number of symbol streams per message and symbol extensions for the communication (including asymmetric complex signaling). Two remarks are elucidated: a) The total DoF of the rank-deficient MIMO X channel can be superior to the full-rank case. Hence, rank-deficiency could be a desired channel property for multipoint-to-multipoint systems, in contrast to point-to-point systems, where rank-deficiency reduces the total DoF. And, b) in terms of DoF, full transmitter cooperation is not always superior to the X channel when channel matrices are low rank.

ACKNOWLEDGMENT

This work has been done in the framework of project TROPIC FP7 ICT-2011-8-318784, funded by the European Community. Also, by the Spanish Science and Technology Commissions and EC FEDER funds through projects: TEC2010-19171/TCM and CONSOLIDER INGENIO CSD2008-00010 COMONSENS, and by project 2009SGR1236 (AGAUR) of the Catalan Administration.

REFERENCES