

Consensus in Correlated Random Wireless Sensor Networks

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Abstract—This contribution studies the convergence of consensus algorithms in random wireless sensor networks with spatially correlated links. Aiming at reducing the convergence time, we adopt an optimization criterion based on the minimization of the spectral radius of a matrix for which we derive closed-form expressions for both directed and undirected topologies. We show that the minimization of the spectral radius assuming constant link weights is a convex optimization problem. The expressions derived subsume known protocols found in literature.

Index Terms—Consensus algorithm, directed topology, probabilistic convergence, random links, spatial correlation, undirected topology, wireless sensor networks.

I. INTRODUCTION

The study of consensus in wireless sensor networks (WSNs) has attracted increased attention in recent years due to a wide range of applications in different fields. In presence of random communication links, the convergence of consensus algorithms is studied in probabilistic terms. Most contributions address the analysis considering statistically independent links existing with a given probability. Related work is found in [1] where convergence in probability is shown using notions of stochastic stability for undirected Erdős-Rényi networks, i.e., bidirectional links with equal probability, and in [2] where the convergence rate is characterized in terms of the eigenvalues of a Lyapunov-like matrix recursion. Assuming different probabilities of connection, [3] relates mean square (m.s.) convergence to the second smallest eigenvalue of the average Laplacian matrix and derives bounds on the convergence rate. Regarding directed random networks, i.e., unidirectional links, and nonnegative stochastic weight matrices, [4] shows almost sure (a.s.) convergence and [5] derives closed-form expressions for the mean and the variance of the consensus value. Admitting weight matrices with negative entries, [6] shows m.s. convergence for directed Erdős-Rényi networks and derives an upper bound for the m.s. error (MSE) for the case of links with different probabilities. Moreover, [7] studies a.s. convergence in continuous systems whereas [8] defines asymptotic m.s. and per-step m.s. convergence factors to characterize the convergence speed of the algorithm.

Communication links may be however spatially correlated not only due to an intrinsic correlation of the channels between pairs of nodes, but also to the communication protocol. An example is the random gossip algorithm, a particular case of consensus with spatially correlated links where a single bidirectional link between two neighboring nodes is established at each iteration [9]. Relevant contributions can be found in [10] which considers the MSE convergence rate as an optimization criterion to assign the weights in undirected networks. For directed networks, [11] derives closed-form expressions for the asymptotic convergence factor assuming fixed out-degree networks. Further-

more, [12] studies the asymptotic convergence rate of randomized protocols and [13] derives a sufficient condition for a.s. convergence of the broadcast gossip algorithm as a particular case of the consensus algorithm in random directed topologies.

This contribution studies the convergence of consensus algorithms in WSNs with both directed and undirected topologies and assuming spatially correlated random links. Whereas [12] and [13] focus specifically on gossip, we consider the more general model in [14], not necessarily restricted to nonnegative symmetric weight matrices. As we will see, the convergence of the consensus algorithm under these connectivity conditions is related to the spectral radius of a positive semidefinite matrix for which we derive a closed-form expression for constant link weights. We consider the minimization of this spectral radius as the optimization criterion to reduce the convergence time of the algorithm and show that the optimum link weight can be obtained as the solution of a convex optimization problem.

The correspondence is organized as follows. Section II summarizes basic concepts of random graphs whereas in Section III we describe the consensus algorithm in random networks and evaluate the convergence conditions of an upper bound. In Section IV we derive closed-form expressions useful to minimize the convergence time of the upper bound and show convexity of the optimization problem. These expressions are further validated by deriving theoretical results of known existing protocols in Section V. Simulation results and conclusions are included in Sections VI and VII respectively.

II. PRELIMINARIES

The information flow among the nodes of an undirected (directed) network with random topology is described by an undirected (directed) graph $\mathcal{G}(k) = \{\mathcal{V}, \mathcal{E}(k)\}$, where \mathcal{V} is the constant set of vertices or nodes and $\mathcal{E}(k)$ is the set of edges or links e_{ij} at time k for all $\{i, j\} \in \{1, \dots, N\}$, such that the information flows from node j to node i [15]. The links exist with a given probability, i.e., $e_{ij} \in \mathcal{E}(k)$ with probability $0 \leq p_{ij} \leq 1$, and in this correspondence the probabilities are assumed symmetric, i.e., $p_{ij} = p_{ji}$ for all $\{i, j\}$. Further, we consider no edges from node i to itself, i.e., $p_{ii} = 0$. The set of neighbors of node i at time k is denoted $\mathcal{N}_i(k) = \{j \in \mathcal{V} : e_{ij} \in \mathcal{E}(k)\}$. A (directed) path is an ordered sequence of vertices such that from each vertex there is an edge to the next vertex in the sequence. A (directed) graph is (strongly) connected if any node can be reached from any other node of the graph by a (directed) path. Let $\mathbf{P} \in \mathbb{R}^{N \times N}$ denote the symmetric connection probability matrix with $\{ij\}^{\text{th}}$ entry $P_{ij} = p_{ij}$. The adjacency matrix $\mathbf{A}(k) \in \mathbb{R}^{N \times N}$ is random and symmetric (nonsymmetric) with temporally independent entries given by $[\mathbf{A}(k)]_{ij} = 1$ with probability p_{ij} , $[\mathbf{A}(k)]_{ij} = 0$ with probability $1 - p_{ij}$, and expected value matrix defined as $\mathbb{E}[\mathbf{A}(k)] = \bar{\mathbf{A}}$ satisfying $\bar{\mathbf{A}} = \mathbf{P}$. The degree matrix $\mathbf{D}(k) \in \mathbb{R}^{N \times N}$ is diagonal with entries $[\mathbf{D}(k)]_{ii} = \sum_{j=1}^N [\mathbf{A}(k)]_{ij}$, and $\mathbf{L}(k) = \mathbf{D}(k) - \mathbf{A}(k)$ is the instantaneous Laplacian matrix. By construction, the smallest eigenvalue of $\mathbf{L}(k)$ in magnitude is $\lambda_N(\mathbf{L}(k)) = 0$ with associated right eigenvector $\mathbf{1} \in \mathbb{R}^{N \times 1}$, an all-ones vector of length N . If $\mathcal{G}(k)$ is (strongly) connected, $\lambda_N(\mathbf{L}(k))$ has algebraic multiplicity one and $\mathbf{L}(k)$ is irreducible. Let \mathbb{S} , \mathbb{S}^+ and \mathbb{S}^- denote the sets of symmetric, positive semidefinite, and negative semidefinite matrices respectively. The expected Laplacian $\bar{\mathbf{L}} = \bar{\mathbf{D}} - \mathbf{P}$ is symmetric by construction and assumed irreducible for both the directed and the undirected case. $\bar{\mathbf{L}} \in \mathbb{S}^+$ and could be seen as the Laplacian of an undirected connected graph $\bar{\mathcal{G}}$, the expected graph of $\mathcal{G}(k)$. The spatial correlation among pairs of links e_{ij} and e_{qr} for nodes i, j, q and r is organized in the matrix $\mathbf{C} \in \mathbb{R}^{N^2 \times N^2}$, defined as follows

$$\mathbf{C}_{st} = \begin{cases} \mathbb{E}[a_{ij}a_{qr}] - p_{ij}p_{qr}, & s \neq t \\ 0, & s = t \end{cases} \text{ for } \begin{cases} s = i + (j - 1)N \\ t = q + (r - 1)N \end{cases} \quad (1)$$

Manuscript received June 11, 2011; revised August 23, 2011; accepted August 23, 2011. Date of publication August 30, 2011; date of current version November 16, 2011. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Dominic K. C. Ho. This work has been partially funded by the Spanish Ministry of Education and Science, FEDER funds (TEC2006-06481 FBNIT, CONSOLIDER CSD2008-00010 COMONSENS and TEC2010-19171 MOSAIC) and the Catalan Government (2009SGR-01236 AGAUR).

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Digital Object Identifier 10.1109/TSP.2011.2166552

with $a_{ij} = [\mathbf{A}(k)]_{ij}$, where the time indexing is omitted since correlation is assumed time invariant.

III. CONSENSUS IN RANDOM NETWORKS

Consider a WSN composed of N nodes indexed with $i \in \{1, \dots, N\}$ and let $\mathbf{x}(k) \in \mathbb{R}^{N \times 1}$ denote the vector of all states at time k , initialized at time $k = 0$ with the values of the measurements. The evolution of $\mathbf{x}(k)$ can be written in matrix form as follows:

$$\mathbf{x}(k+1) = \mathbf{W}(k)\mathbf{x}(k), \quad \forall k > 0 \quad (2)$$

where $\mathbf{W}(k) \in \mathbb{R}^{N \times N}$ is the weight matrix with a nonzero $\{ij\}^{\text{th}}$ entry at time k if node i receives information from node j . The matrices $\{\mathbf{W}(k), \forall k \geq 0\}$ are i.i.d. and have by construction at least one eigenvalue equal to 1 with associated right eigenvector $\mathbf{1}$. For directed topologies $\mathbf{W}(k)$ satisfies for all k : i) $\mathbf{W}(k)\mathbf{1} = \mathbf{1}$, ii.a) $\mathbf{1}^T \mathbf{W}(k) \neq \mathbf{1}^T$ and iii) $\bar{\mathbf{W}}\mathbf{1} = \mathbf{1}$, $\mathbf{1}^T \bar{\mathbf{W}} = \mathbf{1}^T$, whereas for undirected topologies $\mathbf{W}(k)$ satisfies for all k : i), ii.b) $\mathbf{1}^T \mathbf{W}(k) = \mathbf{1}^T$ and iii). A right eigenvector $\mathbf{1}$ in i) implies that after reaching a consensus the network will remain in consensus, and a left eigenvector $\mathbf{1}$ in ii.b) implies that the average of the state vector is preserved from iteration to iteration. In the following section we present a sufficient condition for a.s. convergence in random topologies.

A. Sufficient Condition for a.s. Convergence

Following the line of previous contributions, e.g., [9], [12], [13], we consider the distance to the current average, i.e., the vector of deviations at time k given by

$$\mathbf{d}(k) = (\mathbf{I} - \mathbf{J})\mathbf{x}(k) \quad (3)$$

where \mathbf{I} denotes the $N \times N$ identity matrix and $\mathbf{J} = \frac{\mathbf{1}\mathbf{1}^T}{N}$ is a normalized all-ones matrix. The deviation vector $\mathbf{d}(k)$ specifies how far the nodes are from a consensus at time k . The convergence of $\mathbf{x}(k)$ in (2) to a consensus vector $c\mathbf{1}$ is related to the convergence of the norm of $\mathbf{d}(k)$ in (3) to zero, since

$$\Pr \left\{ \lim_{k \rightarrow \infty} \mathbf{x}(k) = c\mathbf{1} \right\} = \Pr \left\{ \lim_{k \rightarrow \infty} \|\mathbf{d}(k)\|_2^2 = 0 \right\} \quad (4)$$

where $\|\cdot\|$ denotes the 2-norm. Note that the calculation of $\mathbf{d}(k)$ is independent of the consensus value and is easier to analyze than the deviation w.r.t. the average consensus vector given by $\mathbf{x}_{ave} = \mathbf{J}\mathbf{x}(0)$. Using property i) of $\mathbf{W}(k)$ we have

$$\mathbf{d}(k+1) = (\mathbf{I} - \mathbf{J})\mathbf{W}(k)\mathbf{d}(k).$$

Further, the expected norm of $\mathbf{d}(k+1)$ given $\mathbf{d}(k)$ is

$$\mathbb{E} [\|\mathbf{d}(k+1)\|_2^2 | \mathbf{d}(k)] = \mathbf{d}^T(k) \mathcal{W} \mathbf{d}(k) \leq \lambda_1(\mathcal{W}) \|\mathbf{d}(k)\|_2^2$$

where

$$\mathcal{W} = \mathbb{E} [\mathbf{W}(k)^T (\mathbf{I} - \mathbf{J}) \mathbf{W}(k)] \quad (5)$$

and we have used that $(\mathbf{I} - \mathbf{J})$ is symmetric and idempotent. Repeatedly conditioning and replacing iteratively for $\mathbf{d}(k)$ we obtain [9]

$$\mathbb{E} [\|\mathbf{d}(k)\|_2^2] \leq \lambda_1^k(\mathcal{W}) \|\mathbf{d}(0)\|_2^2. \quad (6)$$

Remark that in the undirected case, $\mathbb{E} [\|\mathbf{d}(k)\|_2^2]$ coincides with the MSE of the state, defined as $\text{MSE} = \mathbb{E} [\|\mathbf{x}(k) - \mathbf{x}_{ave}\|_2^2]$. The minimization of $\lambda_1(\mathcal{W})$ in (6) is the optimization criterion chosen to reduce the convergence time of the algorithm.

Lemma 1: The consensus algorithm in (2) with weight matrices satisfying i) and iii) converges almost surely to a consensus value if

$$\lambda_1(\mathcal{W}) < 1 \quad (7)$$

where \mathcal{W} is the matrix defined in (5) and $\lambda_1(\cdot)$ denotes its largest eigenvalue.

Proof: See [8] and [13]. \blacksquare

$\lambda_1(\mathcal{W})$ can be identified as the per-step m.s. convergence factor defined in [8], which sets a sufficient condition for a.s. convergence. From the results in [8, Lemma 2(i)], the per-step convergence factor is greater than the asymptotic convergence factor, which computes the distance to the average vector \mathbf{x}_{ave} . A necessary and sufficient condition for m.s. stability, which is only sufficient for a.s. asymptotic stability [16],¹ is that the asymptotic convergence factor is smaller than 1. Since $\lambda_1(\mathcal{W})$ is greater than the asymptotic convergence factor, $\lambda_1(\mathcal{W}) < 1$ is sufficient to have asymptotic a.s. stability. Therefore, according to [8, Lemma 2(ii)], (7) implies a.s. convergence of the consensus algorithm in (2). The next step consists in analyzing the matrix \mathcal{W} where we focus on constant link weight matrices satisfying i) and iii) with temporally independent but spatially correlated entries.

IV. REDUCING THE CONVERGENCE TIME IN CORRELATED RANDOM NETWORKS

We consider the problem of reducing the convergence time of the consensus algorithm in (2) assuming constant link weights ϵ such that

$$\mathbf{W}(k) = \mathbf{I} - \epsilon \mathbf{L}(k), \quad \forall k \geq 0 \quad (8)$$

where $\mathbf{L}(k)$ is the random Laplacian. Recall that if the instantaneous $\mathcal{G}(k)$ is strongly connected, $\mathbf{L}(k)$ is irreducible and so is $\mathbf{W}(k)$. However, the network may be disconnected for all k but the nodes can still reach a consensus provided only that $\bar{\mathbf{L}}$ is irreducible [1], [3], [4]. Furthermore, to achieve the average consensus $\mathbf{W}(k)$ must satisfy ii.b), or choose a value of ϵ close to zero at the cost of slowing down the convergence speed of the algorithm. Since we focus on minimizing the upper bound in (6), we search for the value of ϵ that minimizes $\lambda_1(\mathcal{W})$. Therefore, we start replacing (8) in (5) such that

$$\mathcal{W} = \mathbb{E} \left[\mathbf{L}(k)^T (\mathbf{I} - \mathbf{J}) \mathbf{L}(k) \right] \epsilon^2 - 2\bar{\mathbf{L}}\epsilon + \mathbf{I} - \mathbf{J}. \quad (9)$$

In the following theorems we provide closed-form expressions for \mathcal{W} in (9) for random networks with correlated communication links, where the weight matrices are assumed symmetric and irreducible in expectation, i.e., $\bar{\mathbf{W}}$ satisfies condition iii) and $\lambda_1(\bar{\mathbf{W}}) = 1$ with algebraic multiplicity one. We consider first the case of directed topologies.

Theorem 1: Consider the consensus algorithm in (2) with N nodes and spatially correlated random links, $\mathbf{W}(k)$ defined in (8) and satisfying i), ii.a) and iii). The matrix \mathcal{W} in (9) has a closed-form expression given by

$$\mathcal{W} = \left(\bar{\mathbf{L}}^2 + \frac{2(N-1)}{N} (\bar{\mathbf{L}} - \hat{\mathbf{L}}) + \mathbf{R} \right) \epsilon^2 - 2\bar{\mathbf{L}}\epsilon + \mathbf{I} - \mathbf{J} \quad (10)$$

where

$$\hat{\mathbf{L}} = \tilde{\mathbf{D}} - \mathbf{P} \odot \mathbf{P} \quad (11)$$

¹In general, necessary and sufficient conditions for stochastic stability can be derived using the results in [17] and [18].

\odot denotes Schur product, $\tilde{\mathbf{D}}$ is diagonal with entries

$$\tilde{\mathbf{D}}_{ii} = [(\mathbf{P} \odot \mathbf{P})\mathbf{1}]_i \quad (12)$$

and $\mathbf{R} \in \mathcal{S}$ is an $N \times N$ matrix built with the covariance terms in \mathbf{C} as follows:

$$\begin{aligned} \mathbf{R}_{mm} &= \mathbf{g}_m^T \mathbf{C} \mathbf{g}_m + \frac{1}{N} (\mathbf{g}_m^T - \mathbf{q}_m^T) \mathbf{C} (\mathbf{q}_m - \mathbf{g}_m) \\ \mathbf{R}_{mn} &= \sum_i \mathbf{e}_{in}^T \mathbf{C} \mathbf{e}_{im} - \mathbf{g}_n^T \mathbf{C} \mathbf{e}_{nm} - \mathbf{g}_m^T \mathbf{C} \mathbf{e}_{mn} \\ &\quad + \frac{1}{N} (\mathbf{g}_n^T - \mathbf{q}_n^T) \mathbf{C} (\mathbf{q}_m - \mathbf{g}_m) \end{aligned} \quad (13)$$

with

$$\mathbf{g}_m = \mathbf{1} \otimes \mathbf{e}_m, \quad \mathbf{q}_m = \mathbf{e}_m \otimes \mathbf{1}, \quad \mathbf{e}_{mn} = \mathbf{e}_m \otimes \mathbf{e}_n, \quad (14)$$

\mathbf{e}_i is the i^{th} column of \mathbf{I} , and \otimes denotes Kronecker product.

Proof: See Appendix A. \blacksquare

We consider now the case of undirected random networks with correlated communication links.

Theorem 2: Consider the consensus algorithm in (2) with N nodes and spatially correlated random links, $\mathbf{W}(k)$ defined in (8) and satisfying i), ii.b) and iii). The matrix \mathcal{W} in (9) has a closed-form expression given by

$$\mathcal{W} = (\tilde{\mathbf{L}}^2 + 2(\tilde{\mathbf{L}} - \tilde{\mathbf{L}}) + \mathbf{R}') \epsilon^2 - 2\tilde{\mathbf{L}}\epsilon + \mathbf{I} - \mathbf{J} \quad (15)$$

where $\tilde{\mathbf{L}}$ is as defined in (11) and

$$\begin{aligned} \mathbf{R}'_{mm} &= \mathbf{g}_m^T \mathbf{C} \mathbf{g}_m \\ \mathbf{R}'_{mn} &= \sum_i \mathbf{e}_{in}^T \mathbf{C} \mathbf{e}_{im} - \mathbf{g}_n^T \mathbf{C} \mathbf{e}_{nm} - \mathbf{g}_m^T \mathbf{C} \mathbf{e}_{mn}, \end{aligned} \quad (16)$$

with \mathbf{g}_m and \mathbf{e}_{mn} defined in (14).

Proof: See Appendix B. \blacksquare

Remark 1: For directed (undirected) networks with uncorrelated random links, a closed-form expression for \mathcal{W} can be obtained considering a zero matrix \mathbf{C} , which makes the term \mathbf{R} in (10) (the term \mathbf{R}' in (15)) vanish.

Notice that the expressions for \mathbf{R} in (13) and \mathbf{R}' in (16) are different. The matrix $\tilde{\mathbf{L}}$ in (11) has the structure of a Laplacian whose off-diagonal entries are the squared entries of $\tilde{\mathbf{A}}$. Therefore, in general $\tilde{\mathbf{L}}$ is not diagonalized by the eigenvectors of $\tilde{\mathbf{L}}$ and a closed-form expression for $\lambda_1(\mathcal{W})$ cannot be derived, except for the special case of links with equal probability of connection.

A. Optimum Link Weights

According to Lemma 1, a.s. convergence of $\mathbf{x}(k)$ in (2) is guaranteed for $\lambda_1(\mathcal{W}) < 1$. Convexity of the optimization function, valid for both undirected and directed topologies, is shown in the following theorem.

Theorem 3: Consider the consensus algorithm in (2) with N nodes, spatially correlated random links, $\mathbf{W}(k)$ defined in (8) satisfying i) and iii), and \mathcal{W} defined in (9) with $\lambda_1(\mathcal{W})$ denoting its largest eigenvalue. The minimization problem

$$\begin{aligned} \min_{\epsilon} \quad & f(\epsilon) = \lambda_1(\mathcal{W}) \\ \text{s.t.} \quad & \epsilon \geq 0, \quad \mathcal{W} \in \mathcal{S}^+ \end{aligned} \quad (17)$$

is convex on ϵ .

Proof: Let the objective function be expressed as $f(\epsilon) = h(G(\epsilon))$, $\text{dom } f = \{\epsilon \in \text{dom } G \mid G(\epsilon) \in \text{dom } h\}$, where $h : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ denotes the maximum eigenvalue function with domain $\text{dom } h = \mathcal{S}$ and the function $\mathcal{W} = G : \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$ is given by

$$G(\epsilon) = \Gamma \epsilon^2 + 2\Delta \epsilon + \Theta \quad (18)$$

with $\Gamma = \mathbb{E}[\mathbf{L}(k)^T (\mathbf{I} - \mathbf{J}) \mathbf{L}(k)]$, $\Delta = -\tilde{\mathbf{L}}$ and $\Theta = \mathbf{I} - \mathbf{J}$, coinciding with (9). We have to show that the composition $f = h \circ G$ is convex, and we do so in three steps: a) show convexity of G , b) show matrix monotonicity of h and c) show convexity of the composition $f = h \circ G$.

a) To show convexity of $G(\epsilon)$ we start showing that $\Gamma \in \mathcal{S}^+$, since for any nonzero $\{\mathbf{v} \in \mathbb{R}^{N \times 1}, \|\mathbf{v}\| = 1\}$ we have

$$\begin{aligned} \mathbf{v}^T \mathbb{E} \left[\mathbf{L}(k)^T (\mathbf{I} - \mathbf{J}) \mathbf{L}(k) \right] \mathbf{v} \\ = \mathbb{E} \left[\mathbf{v}^T \mathbf{L}(k)^T (\mathbf{I} - \mathbf{J})^T (\mathbf{I} - \mathbf{J}) \mathbf{L}(k) \mathbf{v} \right] \geq 0. \end{aligned}$$

Further, observe that for all $x, y \in \text{dom } G$ we have

$$\Gamma (\theta x + (1 - \theta)y)^2 \preceq \Gamma (\theta x^2 + (1 - \theta)y^2)$$

where $\mathbf{A} \preceq \mathbf{B}$ means that $\mathbf{B} - \mathbf{A} \in \mathcal{S}^+$. Adding $2\Delta(\theta x + (1 - \theta)y) + \Theta$ on both sides yields

$$\begin{aligned} \Gamma (\theta x + (1 - \theta)y)^2 + 2\Delta (\theta x + (1 - \theta)y) + \Theta \\ \preceq \theta \Gamma x^2 + (1 - \theta) \Gamma y^2 + 2\Delta x + (1 - \theta) 2\Delta y + \Theta \\ G(\theta x + (1 - \theta)y) \preceq \theta G(x) + (1 - \theta)G(y). \end{aligned}$$

b) Next, we analyze the function h . This function is matrix convex [19], and we will show that it is also matrix monotone. A function $h : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ is matrix monotone w.r.t. the set \mathcal{S} if for any pair $\mathbf{X}, \mathbf{Y} \in \mathcal{S}$, $\mathbf{X} \preceq \mathbf{Y}$ yields $h(\mathbf{X}) \leq h(\mathbf{Y})$. Since the maximum eigenvalue of \mathbf{X} can be seen as the point-wise supremum of a family of linear functions of \mathbf{X} , we have $\lambda_1(\mathbf{X}) = \sup\{\mathbf{u}^T \mathbf{X} \mathbf{u} \mid \|\mathbf{u}\|_2 = 1\}$. Let's denote by \mathbf{u} and \mathbf{v} the eigenvectors associated with the largest eigenvalue of \mathbf{X} and \mathbf{Y} respectively. If $\mathbf{X} \preceq \mathbf{Y}$, for any $\mathbf{u} \in \mathbb{R}^{N \times 1} \Rightarrow \mathbf{u}^T \mathbf{X} \mathbf{u} \leq \mathbf{u}^T \mathbf{Y} \mathbf{u}$. Moreover, since \mathbf{u} is the eigenvector associated to $\lambda_1(\mathbf{X})$, we have $\lambda_1(\mathbf{X}) = \mathbf{u}^T \mathbf{X} \mathbf{u} \leq \mathbf{u}^T \mathbf{Y} \mathbf{u}$. Further, since \mathbf{v} is the eigenvector associated to $\lambda_1(\mathbf{Y})$, we have $\mathbf{u}^T \mathbf{Y} \mathbf{u} \leq \mathbf{v}^T \mathbf{Y} \mathbf{v} = \lambda_1(\mathbf{Y})$, and we can conclude that $\lambda_1(\mathbf{X}) \leq \lambda_1(\mathbf{Y})$.

c) Combining the convexity of G from a) and the matrix monotonicity of h from b) yields

$$h(G(\theta x + (1 - \theta)y)) \leq h(\theta G(x) + (1 - \theta)G(y)) \quad (19)$$

and, recalling the matrix convexity of h , we have

$$h(\theta G(x) + (1 - \theta)G(y)) \leq \theta h(G(x)) + (1 - \theta)h(G(y)). \quad (20)$$

Combining the left-hand side of (19) and the right-hand side of (20) we obtain

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

which completes the proof. \blacksquare

Theorem 3 shows that the function $f(\epsilon)$ is convex. The next step consists in showing that the dynamic range of ϵ for which (7) is satisfied, exists.

B. Existence of Dynamic Range of ϵ

Although $\text{dom } f(\epsilon) = \mathbb{R}^+$, the value of ϵ must be restricted to ensure the convergence of $\mathbf{x}(k)$ in (2). Using (18), $f(\epsilon)$ can be rewritten as follows:

$$\begin{aligned} f(\epsilon) &= \sup_{\|\mathbf{u}\|_2=1} \{\mathbf{u}^T \Gamma \mathbf{u} \epsilon^2 + 2\mathbf{u}^T \Delta \mathbf{u} \epsilon + \mathbf{u}^T \Theta \mathbf{u}\} \\ &= \sup_{\|\mathbf{u}\|_2=1} \{\gamma \epsilon^2 + 2\delta \epsilon + \theta\} \end{aligned} \quad (21)$$

where $\gamma = \mathbf{u}^T \Gamma \mathbf{u}$, $\delta = \mathbf{u}^T \Delta \mathbf{u}$, and $\theta = \mathbf{u}^T \Theta \mathbf{u}$. Observe that Θ has eigenvalues 0 with multiplicity one and 1 with multiplicity $N - 1$. Thus,

for $\epsilon = 0$ we have $\lambda_i(\mathcal{W}) = \lambda_i(\Theta)$ for all i with $\lambda_1(\mathcal{W}) = 1$. For $\epsilon \sim 0$, the quadratic term in (21) is negligible and $\lambda_1(\mathcal{W})$ can be approximated as $f(\epsilon)|_{\epsilon \sim 0} \approx \sup_{\|\mathbf{u}\|_2=1} \{2\delta\epsilon + \theta\}$. For any unitary vector \mathbf{u} we have $0 \leq \theta \leq 1$ and $\delta \leq 0$. Note that $\delta = 0$ if and only if $\mathbf{u} = \mathbf{1}$ and note also that $\mathbf{1}$ is the eigenvector associated with the eigenvalue 0 of Θ . Therefore, there is no \mathbf{u} such that $\delta = 0$ and $\theta = 1$ at the same time, showing that the approximation is always below 1. However, as $\epsilon \rightarrow \infty$ the quadratic term in (21) becomes predominant and $\lambda_1(\mathcal{W}) \rightarrow \infty$. Combining these results with the fact that $f(\epsilon)$ is convex, we conclude that there exists an interval of positive values of ϵ for which (7) holds, ensuring therefore (4). Theorem 3 shows that the optimum ϵ^* minimizing $\lambda_1(\mathcal{W})$ is the solution of a convex optimization problem. This value can be found using the subgradient algorithm [9], [10].

V. PARTICULARIZATION TO EXISTING PROTOCOLS

The expressions in (10) and (15) are particularized for known protocols whose parameters have a closed-form expression (proofs can be found in [20]). This is useful not only to further validate the main results of this contribution, but also to gain insight into the impact of the spatial correlation on the convergence time.

Directed Topologies With Correlated Links: In the broadcast gossip algorithm, at each iteration a single node is randomly chosen with probability $p = \frac{1}{N}$ to transmit its state to the nodes within its connectivity range. The rest remain silent forcing the following correlation among links:

$$\mathbf{C}_{qr}^{ij} = \begin{cases} 0, & j = r \text{ and } i = q \\ p(1-p), & \text{if } j = r, \text{ and } i \neq q \in \mathcal{N}_j \\ -p^2, & \text{if } j \neq r, \text{ and } i \in \mathcal{N}_j \text{ and } q \in \mathcal{N}_r \\ 0, & \text{otherwise} \end{cases} \quad (22)$$

where for simplicity \mathbf{C}_{qr}^{ij} denotes the entry \mathbf{C}_{st} with s and t as defined in (1). Adopting the notation in [13], we let \mathbf{A} , \mathbf{D} and \mathbf{L} denote the adjacency, the degree and the Laplacian matrix of the fixed graph, and $\epsilon = 1 - \gamma$ where γ is the mixing parameter. Then, we have $\tilde{\mathbf{L}} = p\mathbf{L}$ and $\tilde{\mathbf{A}} = p\mathbf{A}$. A closed-form expression for \mathbf{R} in (13) can be found computing the combined covariance sums in (23), (24), (25), and (26). Using the values given in (22), after a series of computations [20], we obtain

$$\mathbf{R} = 2\tilde{\mathbf{L}} - \tilde{\mathbf{L}}^2 + \frac{1}{N} \left(2(\tilde{\mathbf{L}} - \tilde{\mathbf{L}}) - p\tilde{\mathbf{L}}^2 \right)$$

and replacing for \mathbf{R} in (10), we obtain

$$\mathcal{W}_{BG} = (2\tilde{\mathbf{L}} - \tilde{\mathbf{L}}^2)\epsilon^2 - 2\epsilon\tilde{\mathbf{L}} + \mathbf{I} - \mathbf{J}$$

which coincides with the expression for \mathcal{W} in [13, Lemma 2] after replacing for ϵ . Summing up, the expressions for the matrix \mathcal{W} in [13] can be obtained using the general expression in (10), showing that the broadcast gossip algorithm can be treated as a particular case of the consensus algorithm with spatially correlated random links.

Erdős-Rényi Undirected Topologies With Correlated Links: For this example we consider an Erdős-Rényi graph composed of N nodes connected with probability p , and equal covariance $p(1-p)v$ among links with $0 < v < 1$. Assuming an all-ones matrix $\tilde{\mathbf{J}} \in \mathbb{R}^{N^2 \times N^2}$ and identity matrix $\tilde{\mathbf{I}} \in \mathbb{R}^{N^2 \times N^2}$, the matrix in (1) is given by $\mathbf{C} = p(1-p)v(\tilde{\mathbf{J}} - \tilde{\mathbf{I}} - \mathbf{T})$ with $\mathbf{T}_{st} = 1$ for either $s = i + (i-1)N$ or $t = j + (j-1)N$, and $\mathbf{T}_{st} = 0$ otherwise. Since $\mathbf{P}_{ij} = p$ for all $i \neq j$ we have $\tilde{\mathbf{L}} = pN(\mathbf{I} - \mathbf{J})$ and $\tilde{\mathbf{L}} = p^2N(\mathbf{I} - \mathbf{J})$. After computing the correlation sums using (16), we have that $\mathbf{R}' = N(N-2)p(1-p)v(\mathbf{I} - \mathbf{J})$. Therefore, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{L}}$ and \mathbf{R}' are diagonalized by the same set of eigenvectors and using (15) we obtain $\lambda_1(\mathcal{W}) = (Np + (1-p)(2 + v(N-2)))Np\epsilon^2 - 2Np\epsilon + 1$. The optimum ϵ^* solving (17) can be computed taking the derivative of $\lambda_1(\mathcal{W})$ and

solving for ϵ , while the dynamic range is given by $(0, 2\epsilon^*)$, coinciding with the results in [10]. This example is useful to observe that as the correlation increases, the value of $\lambda_1(\mathcal{W})$ increases and the convergence time of the upper bound in (6), defined as

$$\tau = \frac{1}{\log \frac{1}{\lambda_1(\mathcal{W})}},$$

increases as well. In addition, we observe that adding correlation, the dynamic range of ϵ is reduced.

Erdős-Rényi Directed Topologies With Uncorrelated Links: Analogous to the previous case we assume an Erdős-Rényi graph composed of N nodes connected with probability p , but consider instead the matrix \mathcal{W} in (10) with zero matrix \mathbf{R} . The procedure to derive ϵ^* and the dynamic range is similar to the previous one and the results coincide with the expressions derived in [6] minimizing the m.s. convergence rate. Furthermore, we have checked through computer simulations that for links with different probabilities the optimum ϵ^* outperforms the choice of ϵ that minimizes the upper bound for the MSE in [6], resulting in a faster convergence.

VI. SIMULATION RESULTS

We simulate a very general case to support the analytical results: a random geometric network with different probabilities of connection and different correlation among pairs of links. We consider $N = 20$ nodes randomly deployed in a unit square and with fixed position, where two nodes are connected only if the euclidean distance between them is smaller than 0.37. The entries of $\mathbf{x}(0)$ are modeled as Gaussian random variables (r.v.'s) with mean $x_m = 20$ and variance $\sigma_0^2 = 5$ and the links are generated as correlated Bernoulli r.v.'s with different probabilities chosen uniformly between [0, 1]. For the spatial correlation we consider the autoregressive model in [21, Sec. 2.2–2.4] with $p = \max\{\mathbf{P}\}$ and $\theta = 0.3$.

A total of 10 000 independent realizations were run to obtain $\mathbb{E}[\|\mathbf{d}(k)\|_2^2]$, where \mathbf{P} was kept fixed while a new $\mathbf{L}(k)$ was generated at each iteration. Fig. 1 shows the empirical $\mathbb{E}[\|\mathbf{d}(k)\|_2^2]$ in log-linear scale as a function of k for three different values of ϵ :

- 1) $\epsilon = \frac{1}{(N-1)} = 0.0526$ (dotted line);
- 2) $\epsilon_{bound} = 0.1850$, the value minimizing the MSE upper bound defined in [6] (dashed line);
- 3) $\epsilon^* = 0.3367$ minimizing $\lambda_1(\mathcal{W})$ (solid line).

Fig. 2 shows the empirical MSE w.r.t the statistical mean of the initial values in log-linear scale averaged over all nodes for the three cases, along with the benchmark value $\frac{\sigma_0^2}{N}$. The results depicted in Fig. 1 verify that the choice of the optimum ϵ^* reduces the convergence time of the algorithm, whereas the results in Fig. 2 are more useful to evaluate the deviation of the state w.r.t. the statistical mean of the initial measurements caused by asymmetric links.

VII. CONCLUDING REMARKS

The convergence of the consensus algorithm in random WSNs with both directed and undirected topologies and assuming spatially correlated links has been studied, where a useful criterion for reducing the convergence time has been adopted. This criterion is based on the spectral radius of a positive semidefinite matrix for which we derive closed-form expressions for constant link weight matrices, and states a sufficient condition for almost sure convergence. The general expressions provided in the contribution subsume existing protocols found in literature and greatly simplify the derivation of the optimum link weights. The analytical results are further validated with computer simulations of a general case with different probabilities of connection for the links and different correlations among pairs of links. The results obtained for particular cases of links with equal probability of connection

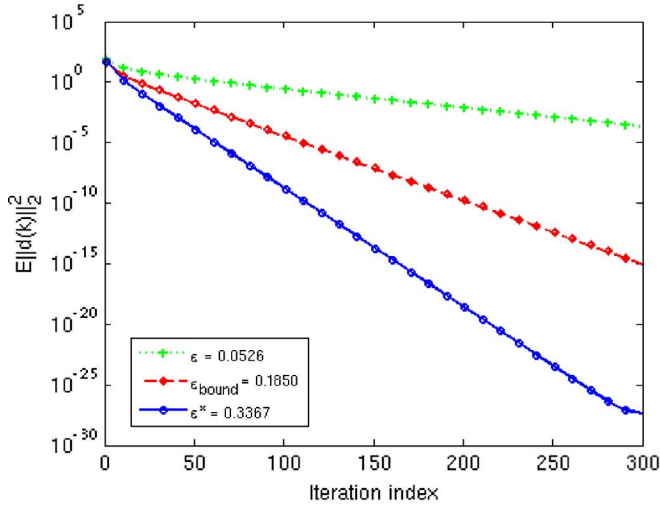


Fig. 1. $\mathbb{E}[\|\mathbf{d}(k)\|_2^2]$ in log-linear scale as a function of the iteration index for different link probabilities and different link weights in a random geometric graph.

show that spatial correlation is detrimental to the convergence rate of consensus algorithms in random topologies and reduces the dynamic range of the link weights.

APPENDIX A PROOF OF THEOREM 1

Recalling that $\mathbf{L}(k) = \mathbf{D}(k) - \mathbf{A}(k)$, at time k we have

$$\begin{aligned} & \mathbb{E}[\mathbf{L}(k)^T \mathbf{L}(k)]_{mm} \\ &= \left(\sum_i a_{mi} \right)^2 + \sum_i a_{im}^2 = \sum_i a_{mi}^2 + \sum_i \sum_{j \neq i} a_{mi} a_{mj} + \sum_i a_{im}^2 \\ & \mathbb{E}[\mathbf{L}(k)^T \mathbf{L}(k)]_{mn} \\ &= -a_{mn}^2 - \sum_{i \neq n} a_{mi} a_{mn} - a_{nm}^2 - \sum_{i \neq m} a_{ni} a_{nm} + \sum_i a_{im} a_{in} \\ & \mathbb{E}[\mathbf{L}(k)^T \mathbf{J} \mathbf{L}(k)]_{mm} \\ &= \frac{1}{N} \left(\left(\sum_i a_{mi} \right)^2 + \left(\sum_i a_{im} \right)^2 \right. \\ & \quad \left. - \sum_i \sum_j a_{mi} a_{jm} - \sum_i \sum_j a_{im} a_{mj} \right) \\ & \mathbb{E}[\mathbf{L}(k)^T \mathbf{J} \mathbf{L}(k)]_{mn} \\ &= \frac{1}{N} \left(\sum_i \sum_j a_{mi} a_{nj} - \sum_i \sum_{j \neq m} a_{im} a_{nj} - \sum_i a_{im} a_{nm} \right. \\ & \quad \left. - \sum_{i \neq n} a_{mi} a_{mn} - a_{mn}^2 + \sum_i a_{im} a_{mn} \right. \\ & \quad \left. - \sum_i \sum_{j \neq m} a_{mi} a_{jn} + \sum_i \sum_{j \neq m} a_{im} a_{jn} \right) \end{aligned}$$

where $a_{mn} = [\mathbf{A}(k)]_{mn}$ and we have considered that $a_{mm} = 0$ for all m . For notation clarity, let \mathbf{C}_{ij}^{qr} denote the entry \mathbf{C}_{st} with s and t as defined in (1). After some manipulations, taking the expectation of the expressions above, we obtain

$$\begin{aligned} & \mathbb{E}[\mathbf{L}(k)^T \mathbf{L}(k)]_{mm} \\ &= \sum_i p_{mi} + \sum_i \sum_{j \neq i} p_{mi} p_{mj} + \sum_i p_{im} + \sum_i \sum_j \mathbf{C}_{mj}^{mi} \end{aligned}$$

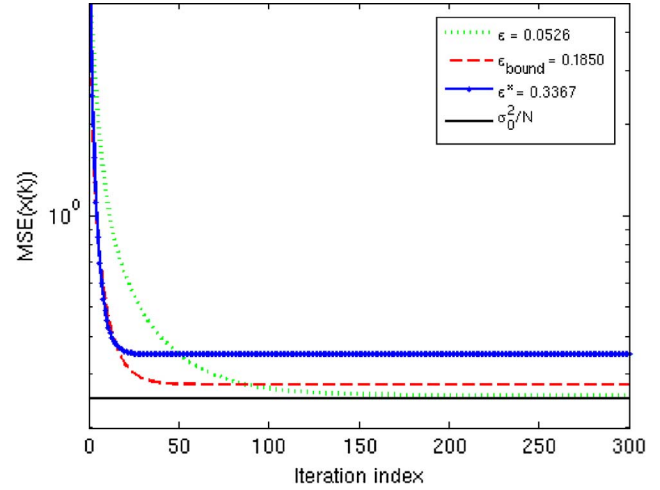


Fig. 2. Empirical $\text{MSE}(x(k))$ in log-linear scale as a function of the iteration index for the same deployment with different link probabilities and different link weights.

$$\begin{aligned} & \mathbb{E}[\mathbf{L}(k)^T \mathbf{L}(k)]_{mn} \\ &= -p_{mn} - \sum_{i \neq n} p_{mi} p_{mn} - p_{nm} - \sum_{i \neq m} p_{ni} p_{nm} \\ & \quad + \sum_i p_{im} p_{in} - \sum_i \mathbf{C}_{mn}^{mi} - \sum_i \mathbf{C}_{nm}^{ni} + \sum_i \mathbf{C}_{in}^{im} \\ & \mathbb{E}[\mathbf{L}(k)^T \mathbf{J} \mathbf{L}(k)]_{mm} \\ &= \frac{1}{N} \left(2 \sum_i p_{mi} + \sum_i \sum_{j \neq i} p_{mi} p_{mj} + \sum_i \sum_{j \neq i} p_{im} p_{jm} \right. \\ & \quad \left. - \sum_i \sum_j p_{mi} p_{jm} - \sum_i \sum_j p_{im} p_{mj} \right. \\ & \quad \left. + \sum_i \sum_j \mathbf{C}_{mj}^{mi} + \sum_i \sum_j \mathbf{C}_{jm}^{im} \right. \\ & \quad \left. - \sum_i \sum_j \mathbf{C}_{jm}^{mi} - \sum_i \sum_j \mathbf{C}_{mj}^{im} \right) \\ & \mathbb{E}[\mathbf{L}(k)^T \mathbf{J} \mathbf{L}(k)]_{mn} \\ &= \frac{1}{N} \left(\sum_i \sum_j p_{mi} p_{nj} - \sum_i \sum_j p_{mi} p_{jn} \right. \\ & \quad \left. - \sum_i \sum_{j \neq m} p_{im} p_{nj} + \sum_i \sum_j p_{im} p_{jn} \right. \\ & \quad \left. - \sum_{i \neq n} p_{im} p_{nm} - \sum_{i \neq n} p_{mi} p_{mn} - p_{mn} - p_{nm} \right. \\ & \quad \left. + \sum_i p_{mi} p_{mn} + \sum_i \sum_j \mathbf{C}_{nj}^{mi} + \sum_i \sum_j \mathbf{C}_{jm}^{im} \right. \\ & \quad \left. - \sum_i \sum_j \mathbf{C}_{nj}^{im} - \sum_i \sum_j \mathbf{C}_{jm}^{mi} \right) \end{aligned}$$

where we have assumed that $\mathbf{C}_{ii} = \mathbf{C}_{ij} = \mathbf{C}_{ij} = 0$, for all $\{i, j, q, r\} \in \{1, \dots, N\}$ and considered that $\mathbf{P} \in \mathbb{S}$ with $\mathbf{P}_{ii} = 0$ for all i . Rearranging terms we have

$$\begin{aligned} & \mathbb{E}[\mathbf{L}(k)^T \mathbf{L}(k)]_{mm} \\ &= 2 \sum_i p_{mi} + \left(\sum_i p_{mi} \right)^2 - \sum_i p_{mi}^2 + \sum_i \sum_j \mathbf{C}_{mj}^{mi} \\ &= 2\bar{\mathbf{D}}_{mm} + \bar{\mathbf{D}}_{mm}^2 - \bar{\mathbf{D}}_{mm} + \sum_i \sum_j \mathbf{C}_{mj}^{mi} \end{aligned} \quad (23)$$

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{L}(k)^T \mathbf{L}(k) \right]_{mn} \\
&= -2p_{mn} - p_{mn} \left(\sum_i p_{mi} + \sum_i p_{ni} \right) + \sum_i p_{im} p_{in} \\
&\quad + 2p_{mn}^2 - \sum_i \mathbf{C}_{mn}^{mi} - \sum_i \mathbf{C}_{mn}^{ni} + \sum_i \mathbf{C}_{in}^{im} \\
&= -2\mathbf{P}_{mn} - [\tilde{\mathbf{D}}\mathbf{P}]_{mn} - [\mathbf{P}\tilde{\mathbf{D}}]_{mn} + [\mathbf{P}^2 - \tilde{\mathbf{D}}]_{mn} \\
&\quad + 2[\mathbf{P} \odot \mathbf{P}]_{mn} - \sum_i \mathbf{C}_{mn}^{mi} - \sum_i \mathbf{C}_{mn}^{ni} + \sum_i \mathbf{C}_{in}^{im} \quad (24)
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{L}(k)^T \mathbf{J}\mathbf{L}(k) \right]_{mm} \\
&= \frac{1}{N} \left(2 \sum_i p_{mi} - 2 \sum_i p_{mi}^2 + \sum_i \sum_j \mathbf{C}_{mj}^{mi} \right. \\
&\quad \left. + \sum_i \sum_j \mathbf{C}_{jm}^{im} - \sum_i \sum_j \mathbf{C}_{jm}^{mi} - \sum_i \sum_j \mathbf{C}_{mj}^{im} \right) \\
&= \frac{1}{N} \left(2\tilde{\mathbf{D}}_{mm} - 2\tilde{\mathbf{D}}_{mm} + \sum_i \sum_j \mathbf{C}_{mj}^{mi} + \sum_i \sum_j \mathbf{C}_{jm}^{im} \right. \\
&\quad \left. - \sum_i \sum_j \mathbf{C}_{jm}^{mi} - \sum_i \sum_j \mathbf{C}_{mj}^{im} \right) \quad (25)
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{L}(k)^T \mathbf{J}\mathbf{L}(k) \right]_{mn} \\
&= \frac{1}{N} \left(-2p_{mn} + 2p_{mn}^2 + \sum_i \sum_j \mathbf{C}_{nj}^{mi} + \sum_i \sum_j \mathbf{C}_{jn}^{im} \right. \\
&\quad \left. - \sum_i \sum_j \mathbf{C}_{nj}^{im} - \sum_i \sum_j \mathbf{C}_{jn}^{mi} \right) \\
&= \frac{1}{N} \left(-2\mathbf{P}_{mn} - [\mathbf{P} \odot \mathbf{P}]_{mn} + \sum_i \sum_j \mathbf{C}_{nj}^{mi} \right. \\
&\quad \left. + \sum_i \sum_j \mathbf{C}_{jn}^{im} - \sum_i \sum_j \mathbf{C}_{nj}^{im} - \sum_i \sum_j \mathbf{C}_{jn}^{mi} \right) \quad (26)
\end{aligned}$$

where $\tilde{\mathbf{D}}$ is the matrix defined in (12), subtracted also in (24) to compensate for the contribution of \mathbf{P}^2 in the main diagonal. Combining the results from (23), (24), (25) and (26) we obtain

$$\begin{aligned}
\mathbb{E} \left[\mathbf{L}(k)^T (\mathbf{I} - \mathbf{J}) \mathbf{L}(k) \right] &= (\tilde{\mathbf{D}} - \mathbf{P})^2 + \frac{2(N-1)}{N} \\
&\quad \times (\tilde{\mathbf{D}} - \mathbf{P} - \tilde{\mathbf{D}} + \mathbf{P} \odot \mathbf{P}) + \mathbf{R} \quad (27)
\end{aligned}$$

where the correlation terms are arranged in the matrix \mathbf{R} as defined in (13). Finally, defining $\tilde{\mathbf{L}}$ as in (11) and replacing (27) in (9) we obtain the closed-form expression in (10), completing the proof.

APPENDIX B PROOF OF THEOREM 2

For the proof of Theorem 2 we can use part of the computations in Appendix A, noting that

$$\mathcal{W} = \mathbb{E} \left[\mathbf{W}(k)^T \mathbf{W}(k) \right] - \mathbf{J} \quad (28)$$

Replacing the weight matrix model (8) in (28), after some matrix manipulations we obtain

$$\mathcal{W} = \mathbb{E} \left[\mathbf{L}(k)^T \mathbf{L}(k) \right] \epsilon^2 - 2\tilde{\mathbf{L}}\epsilon + \mathbf{I} - \mathbf{J} \quad (29)$$

Then, combining the expressions in (23) and (24) yields

$$\mathbb{E} \left[\mathbf{L}(k)^T \mathbf{L}(k) \right] = (\tilde{\mathbf{D}} - \mathbf{P})^2 + 2(\tilde{\mathbf{D}} - \mathbf{P} - \tilde{\mathbf{D}} + \mathbf{P} \odot \mathbf{P}) + \mathbf{R}' \quad (30)$$

where we have arranged the corresponding correlation terms in the matrix \mathbf{R}' , as defined in (16). Finally, replacing (30) in (29) we obtain (15), completing the proof.

ACKNOWLEDGMENT

The authors would like to thank A. Pastore for his useful comments on convexity.

REFERENCES

- [1] Y. Hatano and M. Mesbahi, "Agreement over random networks," in *Proc. 43rd IEEE Conf. Decision Control (CDC)*, Dec. 2004, vol. 2, pp. 2010–2015.
- [2] S. Patterson, B. Bamieh, and A. El Abbadi, "Distributed average consensus with stochastic communication failures," in *Proc. 46th IEEE Conf. Decision Control (CDC)*, 2007, pp. 4215–4220.
- [3] S. Kar and J. Moura, "Distributed average consensus in sensor networks with random link failures," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Apr. 15–20, 2007, vol. 2, pp. II-1013–II-1016.
- [4] A. Tahbaz-Salehi and A. Jadbabaie, "A necessary and sufficient condition for consensus over random networks," *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 791–795, 2008.
- [5] V. M. Preciado, A. Tahbaz-Salehi, and A. Jadbabaie, "Variance analysis of randomized consensus in switching directed networks," in *Proc. Amer. Control Conf. (ACC)*, 2010, pp. 6330–6335.
- [6] S. Silva Pereira and A. Pagès-Zamora, "Mean square convergence of consensus algorithms in random WSNs," *IEEE Trans. Signal Process.*, vol. 58, no. 5, pp. 2866–2874, May 2010.
- [7] M. Porfiri and D. Stilwell, "Consensus seeking over random weighted directed graphs," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1767–1773, 2007.
- [8] J. Zhou and Q. Wang, "Convergence speed in distributed consensus over dynamically switching random networks," *Automatica*, vol. 45, no. 6, pp. 1455–1461, Jun. 2009.
- [9] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2508–2530, 2006.
- [10] D. Jakovetić, J. Xavier, and J. M. F. Moura, "Weight optimization for consensus algorithms with correlated switching topology," *IEEE Trans. Signal Process.*, vol. 58, no. 7, pp. 3788–3801, Jul. 2010.
- [11] N. Abaid and M. Porfiri, "Consensus over numerosity-constrained random networks," *IEEE Trans. Autom. Control*, vol. PP, no. 99, p. 1, Mar. 2010.
- [12] F. Fagnani and S. Zampieri, "Randomized consensus algorithms over large scale networks," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 4, pp. 634–649, 2008.
- [13] T. C. Aysal, M. E. Yildiz, A. D. Sarwate, and A. Scaglione, "Broadcast gossip algorithms for consensus," *IEEE Trans. Signal Process.*, vol. 57, no. 7, pp. 2748–2761, 2009.
- [14] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
- [15] C. Godsil and G. Royle, *Algebraic Graph Theory*. Berlin, Germany: Springer-Verlag, 2001, vol. 207, Graduate Texts in Mathematics.
- [16] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Trans. Autom. Control*, vol. 37, no. 1, pp. 38–53, 1992.
- [17] J. L. Hibey, "Stochastic stability theory for systems containing interval matrices," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 32, no. 4, pp. 1385–1391, 1996.
- [18] K. Benjelloun, E. K. Boukas, and P. Shi, "Robust stochastic stability of discrete-time linear systems with Markovian jumping parameters," in *Proc. 36th IEEE Conf. Decision Control*, 1997, vol. 1, pp. 559–564.
- [19] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.
- [20] S. Silva Pereira, "Distributed consensus algorithms for wireless sensor networks: Convergence analysis and optimization," Ph.D. thesis, Universitat Politècnica de Catalunya-Barcelona Tech, Spain, 2012, to be published.
- [21] A. Lunn and S. J. Davies, "A note on generating correlated binary variables," *Biometrika*, vol. 85, no. 2, pp. 487–490, 1998.