PERFORMANCE ANALYSIS OF SPACE-TIME BLOCK CODING WITH ADAPTIVE MODULATION

Héctor M. Carrasco1, Javier R. Fonollosa2 and J. A. Delgado-Pení3

1,2,3 Universidad Politécnica de Cataluña, C/Jordi Girona 1-3 08034 Barcelona, Spain, 1hector@iis.upc.es, 2feno@gps.iis.upc.es, 3delpen@iis.upc.es

Abstract - We present a general performance analysis of Space-Time Block Coding (STBC) with fixed and variable data rate for wireless communications. The performance is measured in terms of the tradeoff curves between diversity and spatial multiplexing gains, and in terms of symbol error rate (SER) for QAM constellations. Effective data rate and asymptotic approximation of SER expressions for fixed and adaptive modulation are found.

Keywords – Space-Time Block Coding, Diversity, Symbol Error Rate, Adaptive Modulation, Trade-off analysis.

I. INTRODUCTION

In the past few years, a significant progress in code design for transmit diversity over wireless channel has been made. In most practical systems only the receiver knows the channel. For this scenario, Space-Time Block Coding (STBC) and Space-Time Trellis Coding (STTC) have been proposed. These coding techniques permit to obtain significant gains exploiting independent available channels in a multi-input multi-output (MIMO) systems. In general, STTC perform better than STBC, but their decoding complexity tend to be much greater. On the other hand, STBC, developed on the base of a simple scheme for transmission using two transmit antennas [1,2,3], are able to achieve full diversity. Because of the orthogonal design, these coding techniques retain the property of having a very small decoding complexity and asymptotic approximation of SER expressions for fixed and adaptive modulation are found.

This work presents a general performance analysis of STBC for slow and flat fading with fixed and variable size of symbol constellation. The performance is measured in terms of the tradeoff curves between diversity and spatial multiplexing gains, and in terms of symbol error rate (SER) for QAM constellations, which are obtained from a general and compact instantaneous signal to noise power ratio expression (deduced for this work, but demonstration not included for space limitation). Effective data rate and analytical asymptotic approximations of SER expressions for fixed and adaptive modulation are also found.

II. CODE CONSTRUCTION OF STBC

STBC are constructed as P by N transmission matrices \( S_N = \{ s_{p,n} \} \), whose elements are in general linear combinations of signal constellation components taken from an information block of \( K \) independent symbols [2,3]. A total of \( N \) sequences of \( P \times K \) symbols are transmitted simultaneously from each of the \( N \) antennas. The signal observed at each of the \( M \) receive antennas is a linear superposition of the \( N \) transmitted signals, perturbed by noise.

Related transmission schemes include coding matrices for 2 up to 8 transmit antennas and for an arbitrary number of receive antennas, which can make use of complex constellations to increase the spectral efficiency [2,3]. For real constellations, a symbol vector to be transmitted, \( X = [x_1, x_2, \ldots, x_K] \), \( P = K \), is coded as the matrix

\[
S_X = C_X = \{ c_{p,n} \} = \begin{bmatrix}
x_1 & x_2 & \ldots & x_K \\
x_2 & x_3 & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
-x_K & \cdot & \cdot & \ldots & \cdot
\end{bmatrix}
\]  

where \( N \leq K \) and \( c_{p,n} = \delta_{p,n} \delta_{k,p} \). Columns of \( C_X \) are permutations of \( X \) (according to the index \( k(p,n) \) which elements are properly multiplied by \( \delta_{p,n} = \pm 1 \) to provide orthogonal columns.

For \( N=2, 4 \) and 8, the basic square matrices are specified in [2]. For complex constellations, the coding matrix for \( N=2 \) is \( G_2 = [X_1, X_2]^T \cdot [X_2, X_1]^T \). For \( N=4 \) and 8, matrices are constructed from the real matrix \( C_4 \) and \( C_8 \) (with complex symbols), respectively, as \( G_N = [C_N^T, C_N^T] \). The superscript "T" and "*" mean transpose and transpose and conjugated, respectively. For real and complex constellations, and \( N=3 \), the matrices \( C_3 \) and \( G_3 \) are constructed using any 3 columns of \( C_4 \) and \( G_4 \), respectively, for \( N=5, 6 \) and 7, the matrices \( C_5, C_6 \) and \( G_5, G_6 \) are constructed using any 5, 6 and 7 columns of \( C_4 \) and \( G_4 \), respectively. The ratio \( r = K/P \) can be defined as the matrix coding rate, that is the rate of independent symbols transmitted per block from each antenna, \( r \), is directly related with the spectral efficiency of STBC [3]. Two additional special cases for \( N=4 \) are the coding matrices for complex constellations, \( H_4 \) [3], which is a spectral efficient matrix with \( r = 3/4 \) and \( B_4 \) [4], which is a...
sub-optimal matrix in terms of signal to noise ratio gain, with \( r_c = 1 \).

III. GENERAL PERFORMANCE ANALYSIS OF STBC

For flat fading assumption, the MIMO channel is represented by the matrix \( H(t) = [h_{mn}(t)] \), where \( h_{mn}(t) \) is the complex channel gain from the transmit antenna \( m \) to the receive antenna \( n \). For quasi-static fading assumption, i.e. the fading is static over a block of \( K \) symbols, the MIMO channel is represented by the matrix \( H(t) = [h_{mn}] \), where \( h_{mn} \) is the complex channel gain from the transmit antenna \( n \) to the receive antenna \( m \). For quasi-static fading assumption, i.e. the fading is static over a block of \( K \) symbols, the complex baseband receive signal matrix can be written as \( R = [R_1, R_2, \ldots, R_M] = S_t + \Xi \), where \( \Xi = [\Xi_1, \Xi_2, \ldots, \Xi_M] \) is the baseband receive noise matrix, \( R_m = [r_{1,m}, r_{2,m}, \ldots, r_{M,m}]^T \) and \( \Xi_m = [n_{1,m}, n_{2,m}, \ldots, n_{M,m}]^T \). The elements \( n_{m,m} \) are assumed to be independent samples of a zero-mean complex Gaussian random variable with variance \( \sigma_n^2 \). The channel gains \( h_{mn} \) are modeled to provide \( \text{E}[|h_{mn}|^2] = 1 \) and the total signal power is divided into \( N \), so that the expected signal to noise power ratio at each receive antenna is \( \text{SNR} = \sigma_s^2 / \sigma_n^2 \).

Assuming perfect channel state information, we have analyzed and found that in general a linear processing (L) over the receive signal matrix \( R \) can be applied, to obtain the following compact and general expression for the estimation of the transmitted symbol vector

\[
\hat{X} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \cdots & \hat{x}_M \end{bmatrix} = LR \Xi
\]

where \( \gamma = \sum_{n=1}^{N} |h_{mn}|^2 \) and \( \Xi = [\eta_1, \eta_2, \ldots, \eta_M]^T \).

so we can see that STBC can provide diversity gain of order \( NM \) for decoupled received symbols \( x_k \) corrupted by \( \eta_k \), which are linear combinations of the Gaussian noise \( n_{m,m} \). From this result, we have derived that the equivalent instantaneous snr for the received symbols \( x_k \) (excluding the case of matrix \( B_4 \)) can be expressed as

\[
\text{snr} = \frac{\sigma^2}{\sigma_n^2} = \gamma \frac{\text{SNR}}{N, r_c}
\]

In a previous work [5], a particular case of this expression is obtained, but only from simulations. The linear processing to be applied over \( R \) depends on the coding matrix (\( G, B, C_4, G_4, G_8 \) or \( H \); see the references to find the specific coding matrices for each case) as summarized in table 1, where the considered matrices are the following.

\[ \text{i} \) for matrix \( G_2 \) (real and complex constellations, \( N=K=2, P=2, r_c=1 \))

\[
A_n = \begin{bmatrix} h_{1,n} & h_{2,n} \\ h_{2,n} & -h_{1,n} \end{bmatrix} \quad \text{and} \quad R_n = \begin{bmatrix} r_{1,n} \\ r_{2,n} \end{bmatrix}
\]

\[ \text{ii} \) for matrix \( C_4 \) (real constellations, \( N=K=P=4, r_c=1 \))

\[
A_n = \begin{bmatrix} h_{1,n} & h_{1,n} & h_{1,n} & h_{1,n} \\ h_{2,n} & -h_{2,n} & h_{2,n} & -h_{2,n} \\ h_{3,n} & -h_{3,n} & -h_{3,n} & h_{3,n} \\ h_{4,n} & h_{4,n} & -h_{4,n} & -h_{4,n} \end{bmatrix}
\]

\[ \text{iii} \) for matrix \( G_4 \) (complex constellations, \( N=K=P=4, r_c=1 \))

\[
R_n = \begin{bmatrix} r_{1,n} & r_{2,n} & r_{3,n} & r_{4,n} \end{bmatrix} \quad \text{and} \quad R_n^* = \begin{bmatrix} r_{1,n} & r_{2,n} & r_{3,n} & r_{4,n} \end{bmatrix}
\]

Extension of this result can be done for \( C_8 \) (real constellations, \( N=K=P=8, r_c=1 \))

\[ \text{iv} \) for matrix \( G_8 \) (complex constellations, \( N=K=P=8, r_c=1 \))

\[ \text{v} \) for the special (sub-optimal) case of matrix \( B_4 \) (complex constellations, \( N=K=P=4, r_c=1 \))

\[
A_n = \begin{bmatrix} h_{1,n} & h_{1,n} & h_{1,n} + h_{1,n} \\ 0 & 0 & \sqrt{2} \end{bmatrix}
\]

\[
D_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -h_{1,n} & h_{1,n} \\ -h_{1,n} & h_{1,n} & 0 \end{bmatrix}
\]

\[
R_n = \begin{bmatrix} r_{1,n} & r_{2,n} & r_{3,n} & r_{4,n} \end{bmatrix}
\]

in this case, the snr for the received symbols \( x_k \) is

\[
\text{snr} = \frac{\sigma^2}{\sigma_n^2} = \gamma \left( \sum_{m=1}^{M} |r_{m,n}|^2 \right) \frac{\text{SNR}}{4}
\]

Linear processing as stated in \( i, ii, iii \) and \( v \) can be also applied when matrices \( C_6, G_6 \) or \( H_6 \) are used for \( N=3 \) transmit antennas (and \( C_6 \) or \( G_6 \) for \( N=5, 6 \) or \( 7 \)) setting to zero the proper \( h_{m,n} \).
IV. DIVERSITY AND SPATIAL MULTIPLEXING TRADEOFF

Following the pioneering work of [7], a MIMO channel of $NM$ antennas can provide two extreme points of maximal diversity gain $NM$ (full diversity) or maximal spatial multiplexing gain $\min\{N,M\}$. This means that the maximal SNR exponent of the error rate and the maximal number of parallel channels provided by the MIMO channel in order to transmit independent information are obtained, both at high SNR. Given a MIMO channel, both gains can be simultaneously obtained, but there is a fundamental tradeoff between how much of each type of gain can be extracted. In [7] it is established that the limit of the tradeoff curve between diversity gain ($d$) and spatial multiplexing gain ($r$) that any scheme can achieve in the Rayleigh-fading multiple-antenna channel, is an optimal tradeoff curve $d^*(r)$ given by the piecewise-linear function connecting the points $(k,d^*(r))$, $k=0,1,...,\min\{N,M\}$, where $d^*(k)=(N-k)(M-k)$. The tradeoff and optimal tradeoff curves bridge the gap between the two design criteria, that is, diversity and spatial multiplexing gains, however, increasing the diversity advantage comes at a price of decreasing the spatial multiplexing gain, and vice versa.

In addition to providing diversity to improve reliability, multiple-antenna channels can also support higher data rate than single-antenna systems exploiting the spatial parallel channels [7]. Therefore, since the channel capacity increases linearly with $\log$SNR, in order to achieve a certain fraction of the capacity at high SNR, we consider schemes that support a data rate which also increases with $\log$SNR. Let $M_c$ be the size of the constellation which supports $a\log$SNR [b/symbol], with $a$ an arbitrary constant, then the data rate is $R(\text{SNR}) = a \log_a M_c [\text{b/symbol}]$, and thus the spatial multiplexing gain achieved by this scheme at high SNR is $r$. If STBC is used, the effective data rate depends directly of the matrix coding rate $r$, that is $R(\text{SNR}) = a \log_a M_c + r \log_2 \text{SNR} [\text{b/symbol}]$, and the effective spatial multiplexing gain is $r r$.

V. ERROR RATE ANALYSIS

We start deriving the asymptotic symbol error rate of an arbitrary uncoded square $M_c$-QAM constellation for a scheme with one transmit antenna and $L=NM$ receive antennas, using the fact that, from the diversity point of view, STBC are equivalent to a maximal ratio receive combiner (MRRC). Next we extend the results to an adaptive size constellation ($0 < r < 1$) combined with STBC (similar analysis can also be obtained for PSK and non-square QAM constellations but is not included here because of lack of space).

The instantaneous error probability of a $M_c$-QAM symbol is $P_{\text{as}} = 1 - (1 - P_{\text{as}})^{M_c}$, where $P_{\text{as}}$ is the instantaneous error probability of the independent and orthogonal $\sqrt{M_c}$-PAM signals that generate the $M_c$-QAM constellation.

Since the noise is Gaussian, the $\sqrt{M_c}$-PAM instantaneous error probability is given by

$$P_{\text{as}} = \left(1 - \frac{1}{\sqrt{M_c}}\right) \text{erfc} \left(\frac{3/2 \cdot \text{SNR}}{\sqrt{M_c} - 1}\right)$$

where $\text{SNR} = \rho \cdot \gamma$ is the instantaneous signal-noise ratio, $\rho$ is the mean signal-noise ratio received per antenna, and $\gamma = \sum |h_i|^2$. The complex coefficients $h_i$ are normalized so that $\mathbb{E}[|h_i|^2] = 1$.

The symbol error rate is

$$\text{SER} = \mathbb{E}[P_{\text{as}}] = \mathbb{E}\left[1 - (1 - P_{\text{as}})^{M_c}\right]$$

which, for $\rho \to \infty$ ($p_{\sqrt{M_c}} \to 0$), (6) can be approximated by

$$\text{SER} \approx 2 \mathbb{E}\left[p_{\sqrt{M_c}}\right] = 2 \left(1 - \frac{1}{\sqrt{M_c}}\right) \mathbb{E}\left[\text{erfc} \left(\sqrt{\text{SNR} \cdot \gamma}\right)\right]$$

where $\text{SNR}' = \frac{3/2 \cdot \gamma}{\sqrt{M_c} - 1}$.

For complex Gaussian distribution of the coefficients $h_i$ (Rayleigh channel, the typical scenario for practical wireless applications), there is a closed-form solution for (7) [6], which can be expressed as

$$\text{SER} \approx 4 \left(1 - \frac{1}{\sqrt{M_c}}\right)^{L-1} \sum_{k=1}^{L-1} \frac{1}{2} \left(1 - \mu\right) \left(\frac{2}{1 + \mu}\right)^k$$

where $\mu = \frac{\text{SNR}'}{1 + \text{SNR}'}$.

again, for $\text{SNR}' \to \infty$ ($\mu \to 1$), (8) can be approximated by

$$\text{SER} \approx 4 \left(1 - \frac{1}{\sqrt{M_c}}\right)^{L-1} \frac{1}{(4 \cdot \text{SNR}')^L}$$

then

$$\text{SER} \approx 4 \left(1 - \frac{1}{\sqrt{M_c}}\right)^{L-1} \frac{1}{(M_c - 1)^L} \rho^{2L}$$

Table 1

<table>
<thead>
<tr>
<th>$G_{1}, B_{1}$</th>
<th>$G_{2,8}$</th>
<th>$H_{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{n=1}^{N} A_n^R R_n^R$</td>
<td>$\frac{1}{2} \sum_{n=1}^{N} A_n^R R_n^R + A_n^L R_n^L$</td>
<td>$\sum_{n=1}^{N} A_n^R R_n^R + D_n^R R_n^R$</td>
</tr>
</tbody>
</table>

495
Including now a family of adaptive $M_c$-QAM constellations, following size $M_c=\alpha SNR^L$, and STBC where $\rho=SNR/\beta N$, $L=NM$, the expression (10) at high SNR can be simplified to
\[ SER = \alpha SNR^{-\alpha L} \]  
where $\alpha = 4 \left( \frac{\beta N^L}{6} \right)^{NM} \left( \frac{2NM-1}{NM} \right)$

It can be shown that (11) is an asymptotic approximation for the SER at high SNR but also an upper bound, so that $SER = \alpha SNR^{-\alpha L-\beta}$, where $\beta > 0$ and $\beta \rightarrow 0$ when $SNR \rightarrow \infty$, and therefore the approximation (11) can also be used to determine the minimum SNR that provides a certain maximum absolute error when using the approximation.

For the special case of BPSK constellation, there is a closed-form solution for the bit error rate, which can be expressed as
\[ BER = \left[ \frac{1}{2} (1-\mu) \right] \sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right)^{1/2} \left( \frac{1}{2} (1+\mu) \right)^{L-k} \]  
where $\mu = \frac{SNR}{r N + SNR}$

At high SNR, expression (12) can also be asymptotically approximated by
\[ BER = \left( \frac{r N}{4} \right)^{1/2} \left( \frac{2NM-1}{NM} \right) SNR^{-NM} \]  

From the asymptotical results for the SER, it can be concluded that STBC provide full diversity gain of $NM$ for fixed data rate ($r=0$) and diversity gain of $NM(1-r)$ for adaptive modulation with $0 < r < 1$.

Combining the result for the spatial multiplexing gain obtained in section IV and the diversity gain, we can compare the optimal tradeoff curves for a $N\times M$ MIMO channel and for some $N\times M$ STBC examples, as shown in Fig. 1. As it can be seen from the figure, only the scheme of coding matrix $G_2$ with $M=1$ can achieve the optimal diversity gain for any spatial multiplexing gain, the rest of the schemes the spatial multiplexing gain is at most the matrix coding rate. Nevertheless for all cases the maximal diversity gain is optimal for fixed data rate ($r=0$). The limit of spatial multiplexing of orthogonal design is $r=1$, since full rate essentially means that one symbol is transmitted per symbol time.

Figures 2, 3 and 4 show SER vs. SNR performance for some STBC schemes, SER approximation (expression 8) and asymptotic SER (expression 10) for fixed ($r=0$) and asymptotic SER (expression 11) for variable data rate ($r>0$). In figures 2, 3 and 4, the asymptotic SER for variable data rate have been calculated for the parameter $r=0.5, 0.7$ and 0.9, and for the constant $\alpha=0.5, 0.25$ and 0.5, respectively. Fig. 2 shows SER for schemes of coding matrices with coding rate $\frac{1}{2}$, $G_2$ with $M=2$ and $G_4$ with $M=1$, which have exactly the same performance because of the same products $L=NM$ and $rN$. Fig. 3 shows SER for the scheme of coding matrix with coding rate $\frac{1}{4}$, $H_s$ with $M=1$. And Fig. 4 shows SER for a scheme with more diversity gain of matrix with coding rate $\frac{1}{2}$, $G_s$ with $M=2$.
Fig. 3. SER vs. SNR performance of STBC with adaptive modulation for coding matrix $H_4$ with $M=1$. SER approximation (8) and asymptotic SER (10) for fixed data rate (dashed and solid lines, respectively) and asymptotic SER (11) for variable data rate (dotted line).

Fig. 4. SER vs. SNR performance of STBC with adaptive modulation for coding matrix $G_4$ with $M=2$. SER approximation (8) and asymptotic SER (10) for fixed data rate (dashed and solid lines, respectively) and asymptotic SER (11) for variable data rate (dotted line).

It can also be concluded from figures 2, 3 and 4, that despite the spatial multiplexing gain of STBC is limited to the matrix coding rate, the diversity gain can increase with the product $NM$. Therefore, combining STBC with adaptive modulation by increasing the size of the constellation according to SNR, it is possible to increase data rate at the expense of diversity advantages. Increasing the size of the constellation, the data rate increases, but also the SER exponent decreases. Nevertheless, limiting the parameter $r$ to $0 < r < 1$, reliable communication is possible. On the other side, spatial multiplexing and diversity gains can increase by increasing $N$ and/or $M$.

Finally, in any real system, it should be considered that in order to increase the data rate according to logSNR, the size of the constellation can only be integer, so that the asymptotic SER approximation shows the SER tendency of an adaptive QAM modulation by segments of SNR.

VI. CONCLUSIONS

A general performance analysis of STBC has been presented for slow flat fading with fixed and variable size of constellations to support variable data rate. The performance has been evaluated by analyzing the tradeoff curves between diversity and spatial multiplexing gain and in terms of the symbol error rate for square QAM constellations. Effective data rate and analytical asymptotic approximation for high SNR of SER expressions for fixed and adaptive modulation have been derived and evaluated.

As expected STBC achieve full diversity gain but also limited spatial multiplexing gain, however, combining STBC with adaptive modulation by increasing the size of the constellation according to SNR, it is possible to increase data rate at the expense of diversity advantages. Increasing the size of the constellation, the data rate increases, but also the SER exponent decreases. Nevertheless, limiting the parameter $r$ to $0 < r < 1$, reliable communication is possible. On the other side, spatial multiplexing and diversity gains can increase by increasing $N$ and/or $M$.

Finally, in any real system, it should be considered that in order to increase the data rate according to logSNR, the size of the constellation can only be integer, so that the asymptotic SER approximation shows the SER tendency of an adaptive QAM modulation by segments of SNR.

REFERENCES


